DISCRETE TOPOLOGY AND LIMIT POINTS

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The discrete topology on a set X is one in which every point of X is open. Therefore, all subsets under the discrete topology is open. The purpose of this note is to characterize the discrete topology using the notion of limit points.

Definition 0.1. Let X be a topological space. The point $x \in X$ is a **limit point** of $S \subset X$ if every neighborhood of x contains a point $y \in S$ different from x.

Note that it does make sense to speak of a limit point of X considered as a subset of X itself. As a nonexample, consider $X = [0, 1] \cup \{2\}$ with the subspace topology from $\mathbb{R}$. The point $2 \in X$ is not a limit point of X. Let us give a name in recognition of this phenomenon.

Definition 0.2. A point $x \in X$ is an **isolated point** if it is not a limit point of X.

Proposition 0.1. *The singleton $\{x\} \subset X$ is open if and only if x is an isolated point.*

Proposition 0.2. *A topological space X is discrete if and only if every $x \in X$ is an isolated point. In other words, X is discrete if and only if X has no limit points in itself.*

Let X be a topological space. A subset $S \subset X$ is said to be discrete if the subspace topology on S is discrete.

In this situation, S can still have a limit point in X. This does not contradict proposition 0.2 because the proposition only bans limit points within S. As a concrete instance, the set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is a discrete subset of $\mathbb{R}$ with the limit point $0 \in \mathbb{R}$. What we are observing is that discrete subsets are not necessarily closed.

Proposition 0.3. *Let $x \in S \subset X$. Then x is a limit point of S in S (with respect to the subspace topology) if and only if x is a limit point of S in X.*

Proposition 0.4. *The subset $S \subset X$ is discrete if and only if every limit point of S in X lies outside S.*

We give an example where this characterization is useful.

Consider a Riemann surface X. A **divisor** on X is function $D : X \to \mathbb{Z}$ whose support

$$\operatorname{supp} D = \overline{\{x \in X : D(x) \neq 0\}}$$

is discrete, where $\overline{A}$ denotes the closure of A.

The collection $\mathbb{Z}^X$ of all functions from X to $\mathbb{Z}$ is naturally an abelian group with addition given by pointwise addition of functions. Let us show that the collection of divisors forms an abelian group.

The only tricky part is to show that for any two divisors D and D', the set $\operatorname{supp}(D + D')$ is discrete. For any two sets A, B in a topological space, we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Therefore, $\operatorname{supp}(D + D') \subset \operatorname{supp}(D) \cup \operatorname{supp}(D')$.

Since subsets of discrete sets are discrete, we are done if we can show that $\operatorname{supp}(D) \cup \operatorname{supp}(D')$ is discrete. We cannot conclude the discreteness of the union right away because the union of two discrete sets is in general not discrete. Consider the subsets $\{0\}$ and $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ of $\mathbb{R}$. We have to use the fact that both $\operatorname{supp}(D)$ and $\operatorname{supp}(D')$ are closed.

First, an observation which follows easily from proposition 0.4.

Proposition 0.5. *Let A be a closed subset of a topological space X. Then A is discrete if and only if A has no limit point in X.*

Now we can finish the proof that $\operatorname{supp}(D) \cup \operatorname{supp}(D')$ is discrete.

Proposition 0.6. *Let $A, B \subset X$ be discrete closed subset. Then $A \cup B$ is also discrete and closed.*

Proof. For a contradiction, let us suppose that $A \cup B$ has a limit point $x \in X$. Since $A \cup B$ is closed, we may assume without loss of generality that $x \in A$.

Since A is discrete, there is some neighborhood $U \subset X$ of x such that $U \cap A = \{x\}$. Take any neighborhood $V \subset X$ of x. The equalities

$$(V \cap U) \cap (A \cup B) = (V \cap U \cap A) \cup (V \cap U \cap B) = \{x\} \cup (V \cap U \cap B)$$

shows that there must be some $y \in V \cap U \cap B$ different from x. The point y is in both V and B, so it shows that x must be a limit point of B. This contradicts proposition 0.5. □

References

[1] Rick Miranda. **Algebraic curves and Riemann surfaces**, volume 5. American Mathematical Soc., 1995.

[2] James R. Munkres. **Topology**. Prentice Hall, 2nd edition, 2000.