Torsion and complete modules over dg-algebras

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Contents

1 Introduction

Let R be a commutative ring and $I = (x_1, \dots, x_n) \subset R$ a finitely generated ideal. We can study the following two endofunctors on the category of R-modules: the completion functor $\Lambda_I(-) = \varprojlim(R/I^n \otimes_R -)$ and the torsion functor $\Gamma_I(-) = \varinjlim \mathrm{Hom}(R/I^n, -)$.

One can define a certain complex from the sequence $x_1, \dots, x_n$ called the telescope complex. In [4], the local homology of an R-module M is defined as the hom from the telescope complex to M. In the same paper, it is shown that under certain conditions on $x_1, \dots, x_n$ and R, the local homology agrees with the left derived functor $L\Lambda_I$. Similary, the local cohomology is defined as tensoring with the telescope complex and can be shown to agree with the right derived functor $R\Gamma_I$.

Under the condition that $x_1, \dots, x_n$ is weakly pro-regular, [7] showed that the right derived functor $R\Gamma_I$ and the left derived functor $L\Lambda_I$ exhibit an equivalence between two triangulated subcategories of the derived category $D(R)$. These two subcategories are called the category of torsion and complete modules respectively.

More generally, given a compact object A of the derived category $D(R)$, we can define the subcategory of A-torsion and A-complete modules. Then [3] gives an abstract argument which shows that these two subcategories are equivalent.

We begin with some preliminary material on homological algebra for dg-modules over dg-algebras in section 2. Then we give an expository account of the equivalence of torsion and complete modules following [3] in section 3. Finally, in section 4, make some computations over the derived endomorphism algebra to make the equivalence explicit.

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2 Differential homological algebra

The main objects we are concerned with in the present note are dg-modules over dg-algebras. The conventions regarding dg-modules vary a bit from one author to another, so the goal of this section is to establish the notation and terminology we will use. The material in this section, which is completely standard, is based on [1], [12] and [10].

2.1 Basic constructions

Fix k to be a commutative ground ring with 1. We will use homological indexing throughout this note, so our complexes have differentials that decrease degree by 1. Given any complex M and some $m \in M_i$, we will denote the degree i of m by $|m|$.

Definition 2.1. A complex R over k is a **dg-algebra** if there are k-bilinear maps $R_i \times R_j \to R_{i+j}$ for all $i, j \in \mathbb{Z}$ such that $\bigoplus_{n\in\mathbb{Z}} R_n$ becomes an associative algebra with unit over k. Moreover, we require the differentials d to required to satisfy the graded Leibniz rule:

$$d(rr') = d(r)r' + (-1)^{|r|} r d(r')$$

for all $r \in R_i$ and $r' \in R_j$.

Definition 2.2. A complex M over k is a **left dg-R-module** if there are k-bilinear maps $R_i \times M_j \to M_{i+j}$ for all $i, j \in \mathbb{Z}$ such that $\bigoplus_{n\in\mathbb{Z}} M_n$ becomes a graded module over R. Moreover, we require the differentials d to satisfy the graded Leibniz rule:

$$d(rm) = d(r)m + (-1)^{|r|} r d(m)$$

for all $r \in R_i$ and $m \in M_j$.

A right dg-module is similar, except we need to replace the graded Leibniz rule with $d(mr) = d(m)r + (-1)^{|m|} m d(r)$.

Remark 1. The subject of dg-modules is plagued by the number of signs. The organizing principle that we follow here is the Koszul sign rule, which informally says that when we move an element x pass an element y, we should introduce the sign $(-1)^{|x||y|}$. A more rigorous discussion can be found in [10, Tag 0FNG].

Our definitions of dg-modules and dg-algebras take an "external" perspective. This makes connections to homological algebra clearer. For instance, if we let R be the dg-algebra consisting of k in degree 0, then dg-R-modules are exactly the same as complexes over k. Therefore, when we say "complex" in this section, we mean a dg-k-module. We can recover the usual constructions for complexes this way.

However, there is also an "internal" perspective, which views dg-algebra/dg-module as algebra/modules together with the appropriate grading and differentials. The two perspectives are equivalent, but they have different advantages in different situations.

For example, since we already have a good vocabulary for ordinary algebras and modules, we can lift existing definitions to their dg counterparts and ask these definitions to respect the dg structures. Similarly, many basic propositions for algebras and modules can be adjusted for dg-algebras and dg-modules. We will use both perspectives in our exposition, depending on what is more convenient.

Definition 2.3. Let M, M' be left dg-R-modules. A collection of maps $f_n : M_n \to M'_n$ is said to be R-linear if $f_{i+j}(rm) = rf_j(m)$ for all $r \in R_i$ and $m \in M_j$. The R-linearity condition for right dg-modules is $f_{i+j}(mr) = f_j(m)r$. The collection of R-linear map $M \to M'$ will be denoted $\mathrm{Hom}_R(M, M')$.

We will denote the category of left dg-R-modules with morphisms R-linear chain maps by R-**Mod**. The category of right dg-R-modules with R-linear chain maps will be denoted **Mod**-R.

Proposition 2.1. *The categories* R-**Mod** *and* **Mod**-R *are abelian.*

We omit the routine proof. The key point is that the kernels, cokernels and coproduct are computed degreewise.

Definition 2.4. For a dg-R-module M, its **shift** $M[1]$ is as a complex defined by $M[1]_i = M_{i-1}$ with differential $d_{M[1]} = -d_M$. If M is a left dg-module, the R-action on $M[1]$ is given by $r \cdot_{M[1]} m = (-1)^{|r|} rm$. If M is a right dg-module, the R-action on $M[1]$ is given by $m \cdot_{M[1]} r = mr$.

The different signs are needed to make the graded Leibniz rule work.

Definition 2.5. An R-linear map $f : M \to M'$ over a dg-algebra R gives rise to a map $f[1] : M[1] \to M'[1]$ which is simply $f_{n-1} : M[1]_n \to M'[1]_n$ at each degree.

This gives us an automorphism $[1] : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$. We may define for each integer $n \geq 1$ the functor $[n]$ to be the n^{th} fold composite of $[1]$. This definition can be extended in the obvious way to all $n \in \mathbb{Z}$.

Definition 2.6. For dg-R-modules M, M', we define $\mathrm{Hom}_{\mathbf{dg}\text{-}R}(M, M')$ to be the complex where $\mathrm{Hom}_{\mathbf{dg}\text{-}R}(M, M')_i = \mathrm{Hom}_R(M[i], M')$. The differential D is given by $Df = d_{M'} \circ f - (-1)^{|f|} f \circ d_M$.

A map $f \in \mathrm{Hom}_R(M[i], M')$ is called a degree i map from M to M'. The 0-cycles of $\mathrm{Hom}_{\mathbf{dg}\text{-}R}(M, M')$ are precisely the R-linear chain maps. We have a dg-category **dg**-R-**Mod** with objects dg-R-modules and a morphism complex $\mathrm{Hom}_{\mathbf{dg}\text{-}R}(M, M')$ for any pair $M, M' \in$ **dg**-R-**Mod**.

We have the notion of an opposite algebra for dg-algebras, but it has a sign compared to the corresponding notion for ordinary algebras.

Definition 2.7. The **opposite algebra** R^{op} of a dg-algebra R is the dg-algebra which has the same underlying complex and multiplication given by $r \cdot_{R^{op}} r' = (-1)^{|r||r'|} r' r$.

A left dg-R-module M can then be regarded as a right dg-R^{op}-module via the action $m \cdot_{op} r = (-1)^{|r||m|} rm$. A right dg-$R$-module M is similarly a left dg-R^{op}-module via the action $r \cdot_{op} m = (-1)^{|r||m|} mr$.

Definition 2.8. The **tensor product** of a right dg-R-module M and a left dg-R-module M' is the complex $M \otimes_R M'$ whose i^{th} component is $(\bigoplus_{p+q=i} M_p \otimes_k M'_q)/I$ where I is the k-submodule generated by $mr \otimes m' - m \otimes rm'$ for all $m \in M_{p'}$, $m' \in M'_{q'}$ and $r \in R_{l'}$ where $p' + q' + l' = i$. The differential of $M \otimes_R M'$ is

$$D(m \otimes m') = d_M(m) \otimes m' + (-1)^{|m|} m \otimes d_{M'}(m').$$

If R and S are dg-algebras over k, then we can make the complex $R \otimes_k S$ into a dg-algebra by letting $(r \otimes s)(r' \otimes s') = (-1)^{|s||r'|}(rr' \otimes ss')$.

This allows us to define a dg-(R, S)-bimodule to be a left dg-$R \otimes_k S^{op}$-module. The category of dg-(R, S)-bimodule will be denoted R-**Mod**-S or $(R \otimes_k S^{op})$-**Mod**. Here we have R acting on the left and S acting on the right. We also need bimodules where both algebras act on the left. These are dg-$R \otimes S$-modules as usual. The category of such bimodules will be denoted by $(R \otimes_k S)$-**Mod** for clarity.

Definition 2.9. Given an R-linear chain map $f : M \to N$ of dg-modules, we define the **cone** of f to be the dg-module cone(f) which as a graded R module is $N \oplus M[1]$, together with the differential

$$d = \begin{pmatrix} d_N & f \\ 0 & d_{M[1]} \end{pmatrix} : \begin{pmatrix} N_n \\ M_{n-1} \end{pmatrix} \to \begin{pmatrix} N_{n-1} \\ M_{n-2} \end{pmatrix}.$$

We have written out the definitions painstakingly, emphasizing the R-linearity requirements everywhere because this leads to behaviours not observed if we were only working with complexes over k. The other reason is that we settled on how the signs worked, so that in principle every trivial statement can be verified.

2.2 The projective model structure

It is not hard to set up homological algebra in the category R-**Mod** if one has a model structure on it. The basic definitions and propositions regarding model categories can be found in [5] and we use their terminology. In this section, we summarize the relevant definitions and results related to the projective model structure for dg-modules over dg-algebras following [1, §3, §9] and [12, §10, §11].

In order to work with derived categories and derived functors, we need to know the class of weak equivalences in R-**Mod**. It comes as no surprise that they are just the quasi-isomorphisms.

Definition 2.10. An R-linear chain map $f : M \to M'$ of dg-R-modules is a **quasi-isomorphism** if f induces an isomorphism on all homology.

In principle, all one needs to form the derived category are the weak equivalences. To compute derived functor between two categories with weak equivalences, we only need a certain natural weak equivalence, called a "deformation" of the functor. This view of derived categories and derived functors is explained in more details in [9].

Model categories give a nice way to grab hold of the aforementioned deformations and more generally morphisms between objects in the derived category. This is the perspective that we will adopt. We will also mention the concept of "K-projectives", which is a common approach to the derived category.

The proofs of the statements in this section will not be directly useful for the purpose of performing explicit computations and they are a bit long, so we have chosen to omit them. The statements themselves, however, are practical and will be needed.

Let R be a dg-algebra over k. The projective model structure for R-**Mod** is lifted from the projective model structure on chain complexes over k. Before we describe it, let us first make some definitions that are in analogy to topology.

In k-**Mod**, the n-sphere is the dg-k-module $S_k^n = k[n]$ and the $(n+1)$-disk is $D_k^{n+1} = k \xrightarrow{\text{id}} k$ concentrated in degree n and $n+1$. We have an "extension of scalar" functor $F : k\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ given by $F(X) = R \otimes_k X$. It has the forgetful functor $U : R\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod}$ as a right adjoint.

By extension of scalars, we transport the definition of n-spheres and $(n+1)$-disks to R-**Mod** by letting $S_R^n = F(S_k^n)$ and $D_R^{n+1} = F(D_k^{n+1})$.

Theorem 2.2 (Projective model structure)**.** *There is a model structure on R-**Mod** where the weak equivalences are the quasi-isomorphisms and the fibrations are the degreewise surjections. Moreover,*

1. *the collection $I = \{S_R^n \to D_R^{n+1} : n \in \mathbb{Z}\}$ is a set of generating cofibrations;*
2. *the collection $J = \{0 \to D_R^{n+1} : n \in \mathbb{Z}\}$ is a set of generating trivial cofibrations.*

The extension of scalar $F : k\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ is a left Quillen functor.

Now that we have the projective model structure on R-**Mod**, we may form its derived category.

Definition 2.11. The derived category $D(R)$ of R is the homotopy category $\text{Ho}(R\text{-}\mathbf{Mod})$ of R-**Mod** with the projective model structure.

Our exposition gives left modules slightly higher priority, but we will need to talk about right modules. If we want the derived category of right dg-R-modules, we will write $D(R^{op})$.

It is a standard fact that $D(R)$ is triangulated where $M \to N \to L \to M[1]$ is a distinguished triangle if we have a commutative diagram

$$\begin{array}{ccccccc} M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & M[1] \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ M' & \xrightarrow{f} & N' & \longrightarrow & \mathrm{cone}(f) & \longrightarrow & M'[1] \end{array}$$

in $D(R)$. From now on, when we say "triangles", we always mean distinguished triangles for brevity.

The projective model structure for R-**Mod** has two very nice features. Firstly, every object of R-**Mod** in the projective model structure is fibrant. Secondly, the collection of generating cofibrations and generating trivial cofibrations are both reasonable. This makes checking whether an adjunction is Quillen and computing derived functors easy.

We now focus on giving some concrete characterizations of cofibrant objects and cofibrations in R-**Mod**. Note that a dg-algebra R can be regarded as a dg-(R, R)-bimodule. More generally, any shift $R[n]$ of R is a dg-(R, R)-bimodule over itself. Just like ordinary rings, R-linear maps out of a shift of R is determined by where it sends the shift of the identity. Hence, we have the isomorphism $\mathrm{Hom}_{\mathbf{dg}\text{-}R}(R[n], M) \cong M[n]$. in R-**Mod**. This motivates the definition of free dg-modules.

Definition 2.12. A dg-R-module M is said to be **free** if it is isomorphic to a coproduct of shifts of R in R-**Mod**.

For the purpose of homological algebra, this is not the correct notion of freeness. Even in the case $R = k$, free resolutions of ordinary modules are not free in this sense because our definition of freeness forces all differentials to be zero. The "correct" and less restrictive version of freeness for dg-modules is the following.

Definition 2.13. A dg-R-module M is **semi-free** if it admits a semi-free filtration, that is, a filtration $0 = M_0 \subset M_1 \subset M_2 \subset \ldots$ of M in R-**Mod** such that each M_{i+1}/M_i is free and $M = \varinjlim M_i = \bigcup_{i\geq 0} M_i$.

Actually, we may relax the requirement further to obtain the notion of K-projective dg-modules. They are sometimes also called homotopically projective or h-projective. The usual approach to defining left derived functors is to take K-projectives resolutions.

Recall that a dg-R-module M is acyclic if all homology of M vanishes.

Definition 2.14. A dg-R-module P is ***K*-projective** if for every acyclic dg-R-module N, the complex $\mathrm{Hom}_{\mathbf{dg}\text{-}R}(P, N)$ is acyclic in k-**Mod**.

From the viewpoint of model categories, we would like to consider semi-projective dg-modules.

Definition 2.15. A dg-R-module P is **semi-projective** if it is K-projective and also a projective object in R-**Mod**.

Semi-projectiveness have a nice model categorical interpretation.

Theorem 2.3. *The following are equivalent for a dg-R-module M.*

1. *M is cofibrant in the projective model structure.*
2. *M is semi-projective.*
3. *M is a retract of a semi-free dg-R-module.*

This tells us that in particular, we may take semi-free resolutions in order to compute derived functors. This is what we will do in section 4.

Finally, we also have the following characterization of cofibrations in the projective model structure on R-**Mod**.

Theorem 2.4. *A map $f : M \to N$ in R-**Mod** is a cofibration if and only if f is a degreewise injection and $\operatorname{coker} f$ is cofibrant.*

3 Equivalence of torsion and complete modules

We begin with a discussion on derived tensor-hom adjunction in the setting of dg-modules because it is the main technical ingredient used by Dwyer and Greenlees in the proof of their Morita theorem. In the classical Morita theorem, equivalence can only be obtained under some module theoretic finiteness condition. Therefore, we examine in section 3.2 the counterparts to these finiteness conditions in the derived setting. Finally, we give a proof of the Dwyer-Greenlees Morita theorem in section 3.3 once all the tools are in place.

3.1 Derived tensor-hom adjunction

The goal of this section is to review the construction of the derived tensor and hom functors for dg-modules over dg-algebras. The point here is that we will need to work with the derived endomorphism dg-algebra, which is not commutative. Working with bimodules introduces some complications. For example, the derived tensor product does not give us a closed monoidal structure on dg-modules. Thankfully, with some careful bookkeeping, we will see that the problems are not serious.

It is very common to construct the derived tensor product by fixing one argument. This approach is slightly ad hoc. The derived tensor product ought to be a bifunctor from the get-go. To this end, we review the basics of Quillen bifunctors as developed in [5, §4]. Let us first recall what an adjunction of two variables is.

Definition 3.1. Given categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$, an **adjunction of two variables** consists of three functors

$$\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}, \quad \mathrm{Hom}_l : \mathcal{A}^{op} \times \mathcal{C} \to \mathcal{B}, \quad \mathrm{Hom}_r : \mathcal{B}^{op} \times \mathcal{C} \to \mathcal{A},$$

and two natural isomorphisms φ_l, φ_r such that

$$\mathcal{B}(B, \mathrm{Hom}_l(A, C)) \cong_{\varphi_l} \mathcal{C}(A \otimes B, C) \cong_{\varphi_r} \mathcal{A}(A, \mathrm{Hom}_r(B, C))$$

for every $A \in \mathcal{A}$, $B \in \mathcal{B}$, $C \in \mathcal{C}$.

The notion of an adjunction of two variables is obviously modelled after the tensor-hom adjunction. Here is the version of tensor-hom adjunction we will use in this note.

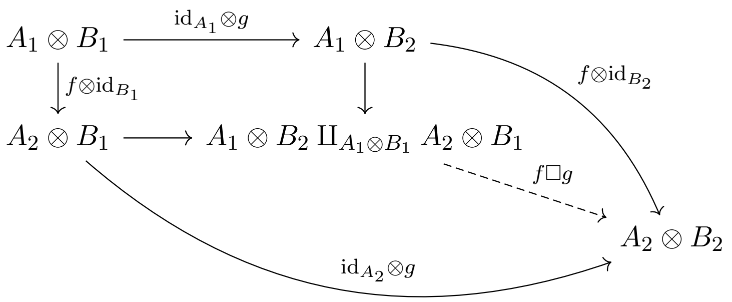
Example 3.1. Suppose that R, S, T are dg-algebras, $M \in R$-**Mod**-S, $N \in S$-**Mod**-T and $L \in R$-**Mod**-T. Then we have an isomorphism

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{dg}\text{-}S \otimes_k T^{op}}(N, \mathrm{Hom}_{\mathbf{dg}\text{-}R}(M, L)) \\ \cong\, &\mathrm{Hom}_{\mathbf{dg}\text{-}R \otimes_k T^{op}}(M \otimes_S N, L) \\ \cong\, &\mathrm{Hom}_{\mathbf{dg}\text{-}R \otimes_k S^{op}}(M, \mathrm{Hom}_{\mathbf{dg}\text{-}T^{op}}(N, L)) \end{aligned}$$

of dg-k-modules that are natural in M, N, L. If we forget about the differentials and the grading, this is just the usual tensor-hom adjunction for modules over algebras. Then one checks that the gradings and differentials are respected by the isomorphisms. We see that $\otimes = \otimes_S, \mathrm{Hom}_l = \mathrm{Hom}_{\mathbf{dg}\text{-}R}, \mathrm{Hom}_r = \mathrm{Hom}_{\mathbf{dg}\text{-}T^{op}}$ is an adjunction of two variables by taking Z_0.

To state the analogue of Quillen adjunctions for adjunctions of two variables, we need the notion of pushout products of morphisms.

Definition 3.2. Given a bifunctor $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, the pushout product of a morphism $f : A_1 \to A_2$ in $\mathcal{A}$ and a morphism $g : B_1 \to B_2$ in $\mathcal{B}$ is the morphism $f \square g$ in the the following pushout diagram in $\mathcal{C}$:



For brevity, we suppress the functors $\mathrm{Hom}_l, \mathrm{Hom}_r$ and the two natural isomorphisms in the definition below.

Definition 3.3. Suppose that $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is an adjunction of two variables between model categories. Let f be a morphism of $\mathcal{A}$ and g be a morphism of $\mathcal{B}$. If the pushout product $f \Box g$ is

1. a cofibration when both f, g are cofibrations,
2. a trivial cofibration whenever f, g are cofibrations and one of f, g is also a trivial cofibration,

then we say that $\otimes$ is a **Quillen adjunction of two variables**.

We will call any of the three functors in a Quillen adjunction of two variables a Quillen bifunctor. The definition seems to be asymmetrical, but we could have equivalently stated everything in terms of Hom_l or Hom_r. This is made precise in [5, 4.2.2].

It is also not hard to see that if $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a Quillen bifunctor and $A \in \mathcal{A}$ is cofibrant, then $A \otimes -$ and $\mathrm{Hom}_l(A, -)$ form a Quillen adjunction pair in the usual sense. Similarly, fixing a cofibrant $B \in \mathcal{B}$ makes $- \otimes B$ into a left Quillen functor.

In our case, it is not difficult at all to check that an adjunction of two variables is actually a Quillen adjunction of two variables. Here is a nice criterion [5, 4.2.5].

Proposition 3.1. *Let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is an adjunction of two variables between model categories. Suppose that I, I' are a set of generating cofibrations for $\mathcal{A}, \mathcal{B}$ respectively, and that J, J' are a set of generating trivial cofibrations for $\mathcal{A}, \mathcal{B}$ respectively. Then $\otimes$ is a Quillen bifunctor if both of the following are satisfied:*

1. *$I \Box I'$ consists of cofibrations,*
2. *$I \Box J'$ and $J \Box I'$ consists of weak equivalences.*

We now check that the adjunction in example 3.1 is a Quillen adjunction of two variables. Our strategy to minimize the computations needed is to check it for the ground ring k first, where it is very easy. Then we use the extension of scalars from the projective model structure of dg-modules to make it work in general.

Proposition 3.2. *Let R, S, T be dg-algebras over k where S is a cofibrant dg-k-module. Then the bifunctor $\otimes_S : R$-**Mod**-$S \times S$-**Mod**-$T \to R$-**Mod**-T is a Quillen bifunctor.*

Proof. Let us first prove the special case when $R = S = T = k$. Without loss of generality, we need to show that if $i : S_k^n \to D_k^{n+1} \in I$, $i' = S_k^m \to D_k^{m+1} \in I'$ and $j' = 0 \to D_k^{m+1} \in J'$, then $i \Box i'$ is a cofibration and $i \Box j'$ is a quasi-isomorphism.

The pushout P of i along i' is concentrated in degree $n+m$ and $n+m+1$. It has a single copy of k at degree $n+m$ and two copies of k at degree $n+m+1$. The pushout product $i \Box i' : P \to D_k^{n+1} \otimes_k D_k^{m+1}$ is injective at each degree and $\operatorname{coker} i \Box i'$ is a single copy of k at degree $n+m+2$, which is cofibrant over $k \otimes_k k^{op} = k$. Here we use the fact that $k^{op} = k$

because k is commutative. By our characterization of cofibrations, we see that $i\square i'$ is a cofibration in k-**Mod**-k.

The pushout product $i\square j'$ is just $S_k^n \otimes_k D_k^{m+1} \to D_k^{n+1} \otimes_k D_k^{m+1}$. Both dg-$k$-modules here are acyclic, so this is a weak equivalence.

For the general case, we need to show that $((R\otimes_k S^{op})\otimes_k i)\square((S\otimes_k T^{op})\otimes_k i')$ is a cofibration and that $((R\otimes_k S^{op})\otimes_k i)\square((S\otimes_k T^{op})\otimes_k j')$ is a weak equivalence.

Note that $S^{op}\otimes_S S = S$ as dg-k-modules. The functors $(R\otimes_k T^{op})\otimes_k - : k\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}\text{-}T$ and $S\otimes_k - : k\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod}$ are left adjoints, so they preserve pushouts. We have proven that $-\otimes_k -$ is a Quillen bifunctor, so our assumption that S is cofibrant over k means that $S\otimes_k -$ is a Quillen functor. We noted in theorem 2.2 that extension of scalar $(R\otimes_k T^{op})\otimes_k -$ is also a Quillen functor. This means that we have

$$((R\otimes_k S^{op})\otimes_k i)\square((S\otimes_k T^{op})\otimes_k i') = (R\otimes_k T^{op})\otimes_k (S\otimes_k (i\square i'))$$

and

$$((R\otimes_k S^{op})\otimes_k i)\square((S\otimes_k T^{op})\otimes_k j) = (R\otimes_k T^{op})\otimes_k (S\otimes_k (i\square j))$$

is a cofibration and a trivial cofibration respectively. $\square$

Exactly like Quillen functors, we can form derived functors from Quillen bifunctors. Let us denote the cofibrant replacement functors by Q and fibrant replacement functors by R.

Definition 3.4. Let $\otimes : \mathcal{A}\times\mathcal{B}\to\mathcal{C}$ be a Quillen bifunctor. Let $A\in \operatorname{Ho}\mathcal{A}$, $B\in\operatorname{Ho}\mathcal{B}$ and $C\in\operatorname{Ho}\mathcal{C}$. The total left derived functor $\otimes^{\mathbb{L}} : \operatorname{Ho}\mathcal{A}\times\operatorname{Ho}\mathcal{B}\to\operatorname{Ho}\mathcal{C}$ is given by $A\otimes^{\mathbb{L}} B = QA\otimes QB$. Similarly, the total right derived functor of $\operatorname{Hom}_l, \operatorname{Hom}_r$ are given by $\mathbb{R}\operatorname{Hom}_l(A,C) = \operatorname{Hom}_l(QA, RC)$ and $\mathbb{R}\operatorname{Hom}_r(B,C) = \operatorname{Hom}_r(QB,RC)$ respectively.

The definition above gives us a recipe for computing derived functors.

Proposition 3.3. *[5, 4.3.1] Suppose that* $\otimes, \operatorname{Hom}_l, \operatorname{Hom}_r$ *is a Quillen adjunction of two variables. Then* $\otimes^{\mathbb{L}}, \mathbb{R}\operatorname{Hom}_l, \mathbb{R}\operatorname{Hom}_r$ *forms an adjunction of two variables between the homotopy categories.*

We will suppress the prefix **dg-** when writing derived hom functors to be in line with standard usage.

Proposition 3.4 (Derived tensor-hom)**.** *Suppose that* R, S, T *are dg-algebras and* S *cofibrant over* k*. Let* $M\in D(R\otimes_k S^{op})$*,* $N\in D(S\otimes_k T^{op})$ *and* $L\in D(R\otimes_k T^{op})$*. Then we have an isomorphism*

$$\mathbb{R}\operatorname{Hom}_{R\otimes_k T^{op}}(M\otimes_S^{\mathbb{L}} N, L)\cong \mathbb{R}\operatorname{Hom}_{R\otimes_k S^{op}}(M, \mathbb{R}\operatorname{Hom}_{T^{op}}(N,L))$$

inside $D(k)$ *natural in* M, N, L*.*

Proof. All functors in the adjunction exist by proposition 3.2 by using suitable dg-algebras. We denote all cofibrant replacement functors by Q. There is no need to talk about fibrant replacements because all objects are fibrant.

Since $- \otimes_S -$ is a Quillen bifunctor, we see that $QM \otimes_S QN$ is also cofibrant. Therefore, we have the natural isomorphisms

$$\begin{aligned} &\mathbb{R}\mathrm{Hom}_{R\otimes_k T^{op}}(M \otimes^{\mathbb{L}}_S N, L) \\ \cong\, &\mathrm{Hom}_{\mathbf{dg}\text{-}R\otimes_k T^{op}}(QM \otimes_S QN, L) \\ \cong\, &\mathrm{Hom}_{\mathbf{dg}\text{-}R\otimes_k S^{op}}(QM, \mathrm{Hom}_{\mathbf{dg}\text{-}T^{op}}(QN, L)) \\ \cong\, &\mathbb{R}\mathrm{Hom}_{R\otimes_k S^{op}}(M, \mathbb{R}\mathrm{Hom}_{T^{op}}(N, L)). \end{aligned}$$

□

We could also specialize the above by taking one or more of the dg-algebras involved to be k. Since the derived tensor-hom adjunction is itself an adjunction on the derived categories when one variable is fixed, we have a unit and counit map associated to it.

Proposition 3.5. *For a fixed $N \in D(S \otimes R^{op})$, we have a unit map*

$$\eta_M : M \to \mathbb{R}\mathrm{Hom}_{R^{op}}(N, M \otimes^{\mathbb{L}}_S N)$$

natural in $M \in D(S^{op})$ called the coevaluation map. We also have a counit map

$$\varepsilon_L : \mathbb{R}\mathrm{Hom}_{R^{op}}(N, L) \otimes^{\mathbb{L}}_S N \to L$$

natural in $L \in D(R^{op})$ called the evaluation map.

3.2 Torsion and complete modules

We begin this section by looking at some standard properties of the triangulated structure on $D(R)$.

Definition 3.5. An object A in a k-linear triangulated category $\mathcal{T}$ is **compact** if $\mathrm{Hom}_{\mathcal{T}}(A, -)$ commutes with all coproducts in $\mathcal{T}$.

The collection of compact objects of a triangulated category $\mathcal{T}$ forms a triangulated subcategory of $\mathcal{T}$ denoted $\mathcal{T}^c$. Note that for $\mathcal{T} = D(R)$ we can equivalently define A to be compact if $\mathbb{R}\mathrm{Hom}_R(A, -) : D(R) \to D(k)$ commutes with all coproducts in $D(R)$.

There are two common types of generation for triangulated categories that we will need here. The first notion is that of a localizing subcategory. Given a full subcategory $\mathcal{S}$ inside a triangulated category $\mathcal{T}$, we write $\mathrm{Loc}\,\mathcal{S}$ for the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{S}$ and is closed under arbitrary coproducts. We call $\mathrm{Loc}\,\mathcal{S}$ the localizing subcategory of $\mathcal{S}$. If $\mathcal{T} = \mathrm{Loc}\,\mathcal{S}$ where $\mathcal{S}$ consists of a set of compact objects, we say that $\mathcal{T}$ is compactly generated and that $\mathcal{S}$ is a set of compact generators for $\mathcal{T}$.

Before we move on to the other type of generation, let us show that $D(R)$ is compactly generated. This is a standard result, which makes certain statements easy to prove if we are able to reduce to checking them for just R. We adapt the proof in [8, §2] to dg-algebras.

Proposition 3.6. *The dg-algebra R regarded as a module over itself is a compact generator for $D(R)$.*

Proof. We first show that R is compact. Let $X \in R$-**Mod**. Note that R is cofibrant and X is fibrant. The homotopy relation $\sim$ on $\mathrm{Hom}_R(R, X)$ coming from the model structure is simply the chain homotopy relation. Therefore, we have

$$H_0(X) \cong H_0(\mathrm{Hom}_{\mathbf{dg}\text{-}R}(R, X)) = \mathrm{Hom}_R(R, X)/\sim = \mathrm{Hom}_{D(R)}(R, X).$$

Since $H_0(-)$ commutes with coproducts, this proves that R is compact.

Now we need to show that $D(R) = \mathrm{Loc}\, R$. Every free dg-R-module is by definition in $\mathrm{Loc}\, R$. Since $\mathrm{Loc}\, R$ is a triangulated subcategory, it is closed under extensions. If $M_0 \to M_1 \to \dots$ is a sequence of morphisms in $D(R)$ where each $M_i \in \mathrm{Loc}\, R$, then we have a triangle of the form

$$\bigoplus_{i=0}^{\infty} M_i \to \bigoplus_{i=0}^{\infty} M_i \to \varinjlim M_i \to \left(\bigoplus_{i=0}^{\infty} M_i\right)[1]$$

which proves that all semi-free dg-R-modules are in $\mathrm{Loc}\, R$.

By an Eilenberg swindle trick, we see that $\mathrm{Loc}\, R$ is closed under summands, so every cofibrant dg-R-module is in $\mathrm{Loc}\, R$ by theorem 2.3. Finally, we have $\mathrm{Loc}\, R = D(R)$ because every dg-R-module has a cofibrant replacement. □

The second type of of generation is thick generation. A strictly full[1] subcategory $\mathcal{U}$ of $\mathcal{T}$ is said to be **thick** if it is a triangulated subcategory closed under taking summands. For a full subcategory $\mathcal{S}$ of $\mathcal{T}$, the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{S}$ is denoted $\mathrm{thick}\, \mathcal{S}$.

Definition 3.6. A dg-module A in $D(R)$ is **perfect** if $A \in \mathrm{thick}(R)$.

There is a well-known characterization which links compact objects, perfect objects and "dualizable" objects in $D(R)$. Let us first define the derived dual object.

Definition 3.7. For a dg-module M in $D(R)$, its dual is $M^\vee = \mathbb{R}\mathrm{Hom}_R(M, R)$ in $D(R^{op})$.

Here $D(R^{op})$ is simply our notation for right dg-R-modules as they are the same as left dg-R^{op}-modules. We put dualizable in quotation marks because of the lack of a closed monoidal structure on $D(R)$.

[1]this means that it is full and also closed under taking isomorphism

Proposition 3.7. *Let R be a dg-k-algebra. The following are equivalent for $A \in D(R)$.*

1. *A is compact.*
2. *A is perfect.*
3. *A is the retract of an $A' \in D(R)$ admitting a finite filtration $0 = A'_0 \subset A'_1 \subset \cdots \subset A'_m = A'$ such that each A'_i/A'_{i-1} is a coproduct of finitely many copies of shifts of R for $1 \leq i \leq m$.*
4. *$A^\vee \otimes^{\mathbb{L}}_R N \cong \mathbb{R}\mathrm{Hom}_R(A, N)$ in $D(k)$ for all $N \in D(R)$.*

We give a proof in the appendix A because it is slightly technical.

For the rest of this section, let us fix a dg-algebra R and compact object $A \in D(R)$. A dg-R-module N is said to be ***A*-trivial** if $\mathbb{R}\mathrm{Hom}_R(A, N) \cong 0$ in $D(k)$.

Example 3.2. [3, 3.1] Let us determine all A-trivial objects when $k = R = \mathbb{Z}$ and $A = \mathbb{Z}/p$, both dg-objects are concentrated in degree 0. Suppose that N is A-trivial. To use the condition $\mathbb{R}\mathrm{Hom}_R(A, N) = 0$, we first need to find a cofibrant replacement for A. In this case, the usual projective resolution $P = \mathbb{Z} \xrightarrow{p} \mathbb{Z}$ concentrated in degree 0 and 1 is a cofibrant replacement for A.

By the definition of $\mathrm{Hom}_{\mathbf{dg}\text{-}R}(P, N)$, we have $\mathrm{Hom}_{\mathbf{dg}\text{-}R}(P, N)_i = N_{i+1} \oplus N_i$. We write elements of $\mathrm{Hom}_{\mathbf{dg}\text{-}R}(P, N)_i = N_{i+1} \oplus N_i$ as column vectors as indicated on the left, then the differential $D : \mathrm{Hom}_{\mathbf{dg}\text{-}R}(P, N)_i \to \mathrm{Hom}_{\mathbf{dg}\text{-}R}(P, N)_{i-1}$ has the matrix representation shown on the right:

$$\begin{pmatrix} N_{i+1} \\ N_i \end{pmatrix} \qquad \begin{pmatrix} d_N & -(-1)^i p \\ 0 & d_N \end{pmatrix} = \begin{pmatrix} d_{\overline{N}[-1]} & (-1)^{i+1} p \\ 0 & -d_{N[-1]} \end{pmatrix}$$

For a dg-k-module M, we let $\overline{M}$ be the dg-k-module given by $\overline{M}_i = M_i$ with differential $d_{\overline{M}} = -d_M$. Clearly $\overline{M}$ and M have the same homology at each degree. If we let $\phi : N \to \overline{N}$ be the chain map where $N_i \to \overline{N}_i$ is given by multiplication by $(-1)^{i+1}p$, then we have $\mathrm{Hom}_{\mathbf{dg}\text{-}R}(P, N) = \mathrm{cone}(\phi[-1])$.

A chain map is a quasi-isomorphism if and only if its mapping cone is acyclic. Therefore, N is A-trivial if and only if multiplication by p is an isomorphism on each homology of N. In other words, N is A-trivial if and only if each homology of N is uniquely p-divisible.

The A-trivial dg-modules are also all those N such that the canonical map $N \to \mathbb{Z}[\frac{1}{p}] \otimes_{\mathbb{Z}} N$ is an isomorphism in $D(R)$. The reason is that $\mathbb{Z}[\frac{1}{p}] \otimes_{\mathbb{Z}} -$ inverts p at each degree of N. Localization commute with taking homology, so

$$H_i\left(\mathbb{Z}\left[\frac{1}{p}\right] \otimes_{\mathbb{Z}} N\right) = H_i(N)\left[\frac{1}{p}\right].$$

Now the canonical map $H_i(N) \cong H_i(N)\left[\frac{1}{p}\right]$ being an isomorphism is the same as saying that $H_i(N)$ is uniquely p-divisible.

Definition 3.8. A dg-R-module X is said to be A-torsion if $\mathbb{R}\mathrm{Hom}_R(X, N) \cong 0$ in $D(R)$ for every A-trivial N.

Definition 3.9. A dg-R-module X is said to be A-complete if $\mathbb{R}\mathrm{Hom}_R(N, X) \cong 0$ in $D(R)$ for every A-trivial N.

The full subcategories of A-torsion and A-complete dg-modules in $D(R)$ will be denoted by $A_{\mathbf{tors}}$ and $A_{\mathbf{comp}}$ respectively.

Proposition 3.8. *The categories $A_{\mathbf{tors}}$ and $A_{\mathbf{comp}}$ are triangulated subcategories of $D(R)$.*

Proof. This requires showing that $A_{\mathbf{tors}}$ and $A_{\mathbf{comp}}$ are additive subcategories of $D(R)$ closed under shifts and taking triangles. For any object $N \in D(R)$, the functors $\mathbb{R}\mathrm{Hom}_R(-, N)$ and $\mathbb{R}\mathrm{Hom}_R(N, -)$ are triangulated. In particular, the proposition follows from taking N to be A-trivial. □

3.3 The Dwyer-Greenlees Morita Theorem

We now follow [3] to prove that the category torsion dg-modules and complete dg-modules are equivalent via a type of Morita theorem.

Let $\mathbb{R}\mathrm{End}_R(M) = \mathbb{R}\mathrm{Hom}_R(M, M)$ for any $M \in D(R)$. In order to compute $\mathbb{R}\mathrm{End}_R(M)$, we take a cofibrant replacement $q : \underline{M} \to M$. Then we have $\mathbb{R}\mathrm{End}_R(M) \cong \mathrm{Hom}_{\mathbf{dg}\text{-}R}(\underline{M}, \underline{M})$ in $D(k)$. In the dg-k-module $\mathrm{Hom}_{\mathbf{dg}\text{-}R}(\underline{M}, \underline{M})$, we can compose a degree i morphism with a degree j morphism to get a degree $i + j$ morphism, so $\mathrm{Hom}_{\mathbf{dg}\text{-}R}(\underline{M}, \underline{M})$ is a dg-algebra.

A natural question is that $\mathbb{R}\mathrm{End}_R(M)$ appears to be dependent on the cofibrant replacement we take for M. One way to proceed is to consider the category **dg-*k*-alg** of dg-k-algebras where the morphisms are ring homomorphisms that are also dg-k-module homomorphisms. We can equip it with the projective model structure. Then one can prove without too much difficulty that different cofibrant replacements of M give rise to isomorphic objects in Ho(**dg-*k*-alg**). The details are worked out in [2, §2.3].

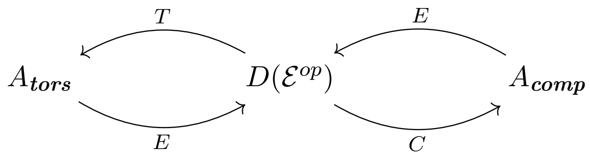
Isomorphic objects of Ho(**dg-*k*-alg**) are connected by a finite zigzag of quasi-isomorphisms in **dg-*k*-alg**. It is also not too difficult to show that quasi-isomorphic dg-k-algebras have equivalent derived categories, so $\mathbb{R}\mathrm{End}_R(M, M)$ is a sensible object to consider.

However, as theorem 3.9 only talks about compact dg-modules, we can sidestep the argument above as follows. The subcategory $M_{\mathbf{tors}}$ of $D(R)$ is independent of the cofibrant replacement for M. Thus, theorem 3.9 can itself be used to justify that the endomorphism dg-algebras obtained from different cofibrant replacements have equivalent derived categories.

Dwyer and Greenlees mention in [3] that their argument can be generalized to many settings with some adjustments. We have made the necessary set up in section 3.1 and 3.2, so now we are in a position to write out what is in our view, a slightly more natural statement for the Morita theorem.

Let R be a dg-algebra and A be a compact object of $D(R)$. Assume that A is cofibrant by fixing a cofibrant replacement and define $\mathcal{E} = \mathbb{R}\mathrm{End}_R(A) \cong \mathrm{Hom}_{\mathbf{dg}\text{-}R}(A, A)$.

Theorem 3.9 (Dwyer-Greenlees)**.** *We have the following equivalences of triangulated categories*



with $E = \mathbb{R}\mathrm{Hom}_R(A, -)$, $T = - \otimes^{\mathbb{L}}_{\mathcal{E}} A$ *and* $C = \mathbb{R}\mathrm{Hom}_{\mathcal{E}^{op}}(A^\vee, -)$.

Before we begin the proof, let us note that we are regarding A as a dg-$R \otimes_k \mathcal{E}$-module and $A^\vee$ as a dg-$R^{op} \otimes_k \mathcal{E}^{op}$-module in the definition of these functors, so we have the correct target categories.

We start with a lemma.

Lemma 3.10. *If $X \to Y$ is a morphism in $A_{\boldsymbol{tors}}$ or $A_{\boldsymbol{comp}}$, then it is an isomorphism if and only if the induced map $E(X) \to E(Y)$ in $D(\mathcal{E}^{op})$ is an isomorphism.*

Proof. We prove the lemma only when $X \to Y$ is in $A_{\mathbf{tors}}$ because the other case is similar. One implication is clear, so we prove that if $E(X) \to E(Y)$ is an isomorphism in $D(\mathcal{E}^{op})$, then $X \to Y$ is an isomorphism in $A_{\mathbf{tors}}$.

Complete $X \to Y$ to a triangle $X \to Y \to Z \to X[1]$. The functor $E = \mathbb{R}\mathrm{Hom}_R(A, -)$ is triangulated, so $E(Z) \cong 0$. This makes Z an A-torsion dg-module that is also A-trivial, which means $\mathbb{R}\mathrm{Hom}_R(Z, Z) \cong 0$. In particular, the map id_Z is zero, so $X \to Y$ is an isomorphism. □

Proof of 3.9. We first check that E has T as a left adjoint and C as a right adjoint. The fact that T is left adjoint to E follows directly from the derived tensor-hom adjunction 3.4. Combining our characterization of compact dg-modules 3.7 with the derived tensor-hom adjunction, we see that C is right adjoint to E.

It follows from adjunctions that if $X \in D(\mathcal{E}^{op})$, then $T(X) = X \otimes^{\mathbb{L}}_{\mathcal{E}} A$ is A-torsion and $C(X)$ is A-complete. This ensures our functors do land in the correct categories. It remains to shows that the unit and counit of the two adjunction pairs are isomorphisms.

The unit $\eta_X : X \to ET(X)$ is an isomorphism when $X = \mathcal{E}$ because

$$ET(\mathcal{E}) = \mathbb{R}\mathrm{Hom}_R(\mathcal{E} \otimes^{\mathbb{L}}_{\mathcal{E}} A, A) = \mathbb{R}\mathrm{Hom}_R(A, A) = \mathcal{E}.$$

The functor T and E preserves coproduct because they are both left adjoints, so the unit $\eta_X : X \to ET(X) = \mathbb{R}\mathrm{Hom}_R(A, X \otimes^{\mathbb{L}}_R A)$ preserves coproducts. Using the five-lemma for triangulated categories and the fact that T and E are triangulated, we see that the collection of $X \in D(\mathcal{E}^{op})$ on which η_X is an isomorphism contains $\mathrm{Loc}\,\mathcal{E}$. This proves that the unit η_X is always an isomorphism.

To see that the counit $\varepsilon_Y : TE(Y) \to Y$ is always an isomorphism, we apply lemma 3.10 which reduce the problem to showing that $E\varepsilon_Y : ETE(Y) \to E(Y)$ is an isomorphism. Since T is left adjoint to E, we have the commutative diagram of natural transformations

$$\begin{array}{ccc} E(Y) & \xrightarrow{\eta_{E(Y)}} & ETE(Y) \\ & \underset{\mathrm{id}_{E(Y)}}{\searrow} & \Big\Downarrow E\varepsilon_Y \\ & & E(Y) \end{array}$$

We already showed that $\eta_{E(Y)}$ is an isomorphism, so we see that $E\varepsilon_Y$ is an isomorphism.

For the counit $\varepsilon'_Z : EC(Z) \to Z$, we first note that $A^\vee \otimes^{\mathbb{L}}_R A \cong \mathcal{E}$ by 3.7. Then the counit being an isomorphism follows from the derived tensor-hom adjunction 3.4.

Finally, to see that unit $\eta'_X : X \to CE(X)$ is an isomorphism, we use lemma 3.10 again and note that we have a commutative diagram of natural transformation

$$\begin{array}{ccc} E(Z) & \xrightarrow{E\eta'_Z} & ECE(Z) \\ & \underset{\mathrm{id}_{E(Z)}}{\searrow} & \Big\Downarrow \varepsilon'_{E(Z)} \\ & & E(Z) \end{array}$$

This is "dual" to the earlier diagram because E is right adjoint to T but left adjoint to C. Since we already proved that the counit $\varepsilon'_{E(Z)}$ is an isomorphism, we see that η'_Z is an isomorphism in $A_{\mathbf{comp}}$. □

We can combine the two equivalences to produce an equivalence of A-torsion dg-modules with A-complete dg-modules.

Corollary. *We have an equivalence of triangulated categories*

$$A_{tors} \underset{(-)^\wedge_A}{\overset{\mathrm{Cell}_A(-)}{\rightleftarrows}} A_{comp}$$

with $(-)^\wedge_A = \mathbb{R}\mathrm{Hom}_R(A^\vee \otimes^{\mathbb{L}}_{\mathcal{E}} A, -)$, $\mathrm{Cell}_A(-) = (A^\vee \otimes^{\mathbb{L}}_{\mathcal{E}} A) \otimes^{\mathbb{L}}_R -$.

Proof. The functor $(-)^\wedge_A$ is obtained directly by applying the derived tensor-hom adjunction to the functor CE in theorem 3.9.

Now, we apply proposition 3.7 applied to the functor TE from theorem 3.9. Using the fact that the derived tensor is associative, we have the isomorphism

$$(A^\vee \otimes^{\mathbb{L}}_R Z) \otimes^{\mathbb{L}}_{\mathcal{E}} A \cong A \otimes^{\mathbb{L}}_{\mathcal{E}^{op}} (A^\vee \otimes^{\mathbb{L}}_R Z) \cong (A \otimes^{\mathbb{L}}_{\mathcal{E}^{op}} A^\vee) \otimes^{\mathbb{L}}_R Z \cong (A^\vee \otimes^{\mathbb{L}}_{\mathcal{E}} A) \otimes^{\mathbb{L}}_R Z$$

natural in $Z \in A_{\mathbf{comp}}$. We are just using the fact that we may consider a left/right dg-$\mathcal{E}$-module as a right/left dg-$\mathcal{E}^{op}$-modules.

This gives the functor $\mathrm{Cell}_A(-) \cong CE$. □

There are two reasons for rephrasing our equivalence in terms of the functors $\mathrm{Cell}_A(-)$ and $(-)^\wedge_A$. The first reason is that $\mathrm{Cell}_A(-)$ and $(-)^\wedge_A$ are actually functor defined on $D(R)$ which give the best A-torsion and A-complete approximations. If R is a commutative ring and I is a finitely generated ideal of R, it is shown in [3] that $\mathrm{Cell}_{R/I}(-)$ computes the local cohomology of an R-module at I. Similarly, the local homology is computed by $(-)^\wedge_{R/I}$. This relates to the background we mentioned in the introduction of this note.

The second reason is that this slight reformulation also highlights the role played by the dg-R-bimodule $A^\vee \otimes^{\mathbb{L}}_{\mathcal{E}} A = \mathrm{Cell}_A(R)$ in the equivalence between $A_{\mathbf{tors}}$ and $A_{\mathbf{comp}}$. This means that in order to make explicit computations, we should understand what $A^\vee \otimes^{\mathbb{L}}_{\mathcal{E}} A$ looks like. When R is commutative and I is a finitely generated ideal, the module $\mathrm{Cell}_{R/I}(R)$ is identified with the telescope complex by an indirect argument in [3].

The next section gives a cofibrant model for $A^\vee \otimes^{\mathbb{L}}_{\mathcal{E}} A$ making use of the Morita theorem and describes it when A is a Koszul complex.

4 Explicit resolutions

In this section, we recall the process of totalizing a chain complex of dg-modules. Then we apply this construction to obtain explicit cofibrant models for $A^\vee \otimes^{\mathbb{L}}_{\mathcal{E}} A$ when A is a Koszul complex under certain conditions on R.

4.1 Totalization

Our convention for double complexes is that they have commuting differentials. An appropriate sign must be introduced in the differential of the total complex.

Definition 4.1. Suppose that we have a chain complex

$$M = \ldots \longleftarrow M_{-1} \xleftarrow{d^h} M_0 \xleftarrow{d^h} M_1 \longleftarrow \ldots$$

in R-**Mod**. Let $M_{p,q}$ be the degree q component of M_p. We define the (direct sum) **totalization** of M to be $\mathrm{Tot}^{\oplus} M$ whose degree n component is $\bigoplus_{p+q=n} M_{p,q}$. The differential of $\mathrm{Tot}^{\oplus}$ on the component $M_{p,q}$ is $d = d^h + (-1)^p d_{M_p}$. If $r \in R_n$ and $x \in M_{p,q}$, then the R-action is given by $r \cdot x = (-1)^{pn} rx$.

The sign for the R-action is so that the Leibniz rule holds for $\mathrm{Tot}^{\oplus} M$. It can be obtained by thinking of $\mathrm{Tot}^{\oplus} M = \bigoplus M_p[-p]$ as a graded module. Now we describe the main tool we will use to construct a semi-free resolution.

Proposition 4.1. *Suppose that we have an acyclic chain complex*

$$M = \ldots \longleftarrow 0 \longleftarrow M_{-1} \xleftarrow{d^h} M_0 \xleftarrow{d^h} M_1 \longleftarrow \ldots$$

*in R-**Mod** such that*

$$\ldots \longleftarrow 0 \longleftarrow Z(M_{-1}) \xleftarrow{d^h} Z(M_0) \xleftarrow{d^h} Z(M_1) \longleftarrow \ldots$$

is also acyclic. Here $Z(M_i)$ denote the dg-module consisting of the cycles of M_i with respect to d_{M_i} and all differentials zero. Then we have a quasi-isomorphism $\mathrm{Tot}^{\oplus} M' \simeq M_{-1}$ where

$$M' = \ldots \longleftarrow 0 \longleftarrow M_0 \xleftarrow{d^h} M_1 \longleftarrow \ldots$$

Although the proof is quite standard, we give the details to indicate why the assumption on cycles are needed. The proof requires a lemma. We keep the notations of proposition 4.1.

Lemma 4.2. *As dg-R-modules, we have*

$$(\mathrm{Tot}^{\oplus} M)[1] = \mathrm{cone}(\rho : \mathrm{Tot}^{\oplus} M' \to M_{-1}).$$

Proof. Let $\rho : \mathrm{Tot}^{\oplus} M' \to M_{-1}$ be given on $M_{p,q}$ by $-d^h$ if $p = 0$ and the zero map if $p \neq 0$. We omit the simple verification that ρ is an R-linear chain map.

For a fixed integer n, we have

$$(\mathrm{Tot}^{\oplus} M)[1]_n = M_{-1,n} \oplus \bigoplus_{\substack{p+q=n-1 \\ p \geq 0}} M_{p,q} = M_{-1,n} \oplus (\mathrm{Tot}^{\oplus} M')[1]_n.$$

Unpacking the definitions, we see that $\mathrm{cone}(\rho)$ has the same differential as $(\mathrm{Tot}^{\oplus} M)[1]$ and R-action, so we are done. ☐

Now we prove proposition 4.1 following [6]. This is done using a truncation trick, so recall that for a chain complex C, the *good* truncation $\tau_{\leq q} C$ is the complex defined by

$$(\tau_{\leq q} C)_n = \begin{cases} C_n & \text{if } n > q; \\ Z_q(C) & \text{if } n = q; \\ 0 & \text{if } n < q. \end{cases}$$

In other words, the good truncation cuts off everything below degree q but keeps the homology at degree q intact.

Proof of 4.1. By lemma 4.2, it suffices to show that $\mathrm{Tot}^{\oplus} M$ is acyclic. Since the R-module structure on each M_i is irrelevant to showing that $\mathrm{Tot}^{\oplus} M$ is acyclic, we can simply consider M as a double complex of k-modules. Let us rename the differentials d_{M_i} to be d^v because they are simply the vertical differentials of the double complex M. It will not be important to us which column they are in.

The chain complexes $\tau_{\leq q}(M_i)$ of good truncations forms a double complex $\tau_{\leq q}M$ whose qth row consists of the qth cycle of each M_i. This means that the double complex M admits a filtration $\tau_{\leq q}M \hookrightarrow \tau_{\leq q-1}M$ such that $\varinjlim \tau_{\leq q}M = M$.

For any abelian category $\mathcal{A}$, we denote its category of chain complexes by $\mathrm{Ch}(\mathcal{A})$. We have a category $\mathrm{Ch}(k\text{-}\mathbf{Mod})$ of double complexes of k-modules on which $\mathrm{Tot}^{\oplus} : \mathrm{Ch}(k\text{-}\mathbf{Mod}) \to k\text{-}\mathbf{Mod}$ is a functor. Thus, we have a canonical map $\varinjlim \mathrm{Tot}^{\oplus} \tau_{\leq q}M \to \mathrm{Tot}^{\oplus} \varinjlim \tau_{\leq q}M = \mathrm{Tot}^{\oplus} M$. This map is clearly an isomorphism at each degree, so it is an isomorphism. Since homology commutes with direct limits, we have reduced the problem to showing that $\mathrm{Tot}^{\oplus} \tau_{\leq q}M$ is acyclic.

The double complex $\tau_{\leq q}M$ is a first quadrant double complex, so there is a spectral sequence

$$E^2_{k,l} = H_k(H_l(\tau_{\leq q}M, d^h), d^v) \Rightarrow H_{k+l}(\mathrm{Tot}^{\oplus} \tau_{\leq q}M)$$

obtained by filtering the rows. This means that we compute $E^2_{k,l}$ by first taking homology with respect to d^h. The chain complex of cycles

$$\dots \longleftarrow 0 \longleftarrow Z(M_{-1}) \xleftarrow{d^h} Z(M_0) \xleftarrow{d^h} Z(M_1) \longleftarrow \dots$$

are by assumption acyclic, so every row of $\tau_{\leq q}M$ is exact. This proves that $\mathrm{Tot}^{\oplus} \tau_{\leq q}M$ is acyclic as needed. □

Of course one could replace the spectral sequence argument above by a diagram chase, but in this case, the spectral sequence argument is very easy and also explains the assumption of proposition 4.1. If instead of a first quadrant double complex, we had a right half-plane double complex, then the spectral sequence would converge weakly to the direct *product* totalization instead [11, 5.6.2]. This is why we needed to truncate and assume that the cycles form acyclic complexes.

4.2 Resolutions for Koszul complexes

Let R be an ordinary commutative ring, so that it is concentrated in degree 0. The goal of this section is to find an explicit cofibrant model for $A^{\vee} \otimes^{\mathbb{L}}_{\mathcal{E}} A$ when A is a length 1 Koszul complex. We will make some comments on Koszul complexes of longer length afterwards.

Let $A = K(x) = \ Re_x \xrightarrow{x} R1$ with e_x being the generator of the degree 1 component. The endomorphism algebra $\mathcal{E}$ is the dg-algebra

$$\mathrm{End}_R(A) = \mathrm{Hom}_{\mathbf{dg}\text{-}R}(K(x), K(x)) = \ R \xrightarrow{\begin{pmatrix} x \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & -x \end{pmatrix}} R$$

concentrated in degree $-1, 0, 1$. If we represent elements of A as column vectors where the degree 1 element is in the top row and the degree 0 element is in the bottom row, then we may represent elements of $\mathcal{E}$ as 2×2 matrices over R. In a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $\mathcal{E}$, the element b corresponds to a map in $\mathrm{Hom}_R(R1, Re_x)$, so it has degree 1. Similarly, the element a corresponds to a map in $\mathrm{Hom}_R(Re_x, Re_x)$, so it has degree 0. The element d corresponds to a map in $\mathrm{Hom}_R(R1, R1)$, so it also has degree 0. Finally, the element c corresponds to a map $\mathrm{Hom}_R(Re_x \to R1)$, so it has degree -1. In our matrix notation, maps of the same degree live in the same diagonal.

The sign in the action of shifts is a slightly subtle point, so let us work that out first. Suppose that $\mathcal{E}[i]$ is regarded as a *left* dg-$\mathcal{E}$-module and $L \in \mathcal{E}[i]$. For any $U = \begin{pmatrix} u & v \\ w & z \end{pmatrix} \in \mathcal{E}$, put $U' = \begin{pmatrix} u & -v \\ -w & z \end{pmatrix}$. The action of $\mathcal{E}$ on $\mathcal{E}[i]$ is given by

$$U \cdot_{\mathcal{E}[i]} L = \begin{cases} UL & \text{if } i \text{ is even;} \\ U'L & \text{if } i \text{ is odd.} \end{cases}$$

Here UL and $U'L$ are understood to be matrix multiplication.

Let $M_0 = \mathcal{E}$ as a left dg-$\mathcal{E}$-module. Since $Z_0(\mathrm{Hom}_{\mathbf{dg}\text{-}\mathcal{E}}(M_0, A)) = Z_0(A) = R$, we can define a map $d_0^h : M_0 \to A$ by $d_0^h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. For each $i \geq 1$, we define $M_i = \mathcal{E}[-i]$. Now

$$Z_0(\mathrm{Hom}_{\mathbf{dg}\text{-}\mathcal{E}}(M_i, M_{i-1})) = Z_{-1}(\mathcal{E}) = R,$$

so we pick the $\mathcal{E}$-map d_i^h given by the element $1 \in R$. This map can be described as right multiplication by the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. More explicitly, due to the sign involved, for $i \geq 1$ we actually have

$$d_i^h \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -b & 0 \\ d & 0 \end{pmatrix}.$$

A straightforward linear algebra computation shows that $\operatorname{im} d^h_{i+1} = \ker d^h_i$ for all $i \geq 0$, so we do have an acyclic complex of dg-$\mathcal{E}$-modules

$$M = \ldots \longleftarrow 0 \longleftarrow A \xleftarrow{d^h} M_0 \xleftarrow{d^h} M_1 \longleftarrow \ldots$$

Let ${}_xR$ be all elements of R annihilated by x and T the R-submodule of R^2 consisting of all elements (a, d) such that $a - d \in {}_xR$. Then the cycle dg-module $Z(A)$ is $Z(A) = R \xleftarrow{0} {}_xR$ in degree 0 and 1. We also have $Z(\mathcal{E}) = R \xleftarrow{0} T \xleftarrow{0} {}_xR$ concentrated in degree $-1, 0, 1$. It again follows from straightforward linear algebra that

$$\ldots \longleftarrow 0 \longleftarrow Z(A) \xleftarrow{d^h} Z(M_0) \xleftarrow{d^h} Z(M_1) \longleftarrow \ldots$$

is an acyclic complex of dg-modules. Since the condition for proposition 4.1 is satisfied, the left dg-$\mathcal{E}$-module

$$\underline{A} = \operatorname{Tot}^{\oplus}(\ \ldots \longleftarrow 0 \longleftarrow M_0 \xleftarrow{d^h} M_1 \longleftarrow \ldots\ ) \qquad (*)$$

is quasi-isomorphic to A. By construction, the dg-$\mathcal{E}$-module admits the filtration $F_0 \hookrightarrow F_1 \hookrightarrow \ldots$ where

$$F_i = \operatorname{Tot}^{\oplus}(\ \ldots \leftarrow 0 \leftarrow M_0 \xleftarrow{d^h} \ldots \xleftarrow{d^h} M_i \leftarrow 0 \leftarrow \ldots\ )$$

We clearly have $\varinjlim F_i = \underline{A}$ and $F_{i+1}/F_i = M_{i+1}$ is free, so $\underline{A}$ is in fact semi-free. This proves that $\underline{A}$ is a cofibrant replacement for A in $\mathcal{E}$-**Mod**. We can organize the result slightly better.

Consider the R-module $M_2(R[t])$ of 2×2 matrices over the polynomial ring of one variable t with coefficients in R. Elements of $M_2(R[t])$ are represented as $\begin{pmatrix} P(t) & Q(t) \\ V(t) & W(t) \end{pmatrix}$. We endow $M_2(R[t])$ with a left dg-$\mathcal{E}$-module structure as follows. Give the elements corresponding to Q degree 1, the elements corresponding to P, W degree 0 and the elements corresponding to V degree -1. It also has a natural left $\mathcal{E}$-action by left multiplication.

If $Z(t) = r_n t^n + \cdots + r_0 \in R[t]$ is any polynomial, denote by $\overline{Z}$ the polynomial $r_n t^{n-1} + \cdots + r_1$. If $\deg Z = 0$, we let $\overline{Z} = 0$. For a fix $x \in R$, we equip $M_2(R[t])$ with the differential

$$d_x \begin{pmatrix} P(t) & Q(t) \\ U(t) & W(t) \end{pmatrix} = \begin{pmatrix} xQ(t) - \overline{Q}(t) & 0 \\ \overline{W}(t) + x(P(t) - W(t)) & xQ(t) \end{pmatrix}$$

Let $\underline{A}$ be the cofibrant replacement of $A = K(x)$ over $\mathcal{E}$ as in *.

Proposition 4.3. *We have* $\underline{A} \cong (M_2(R[t]), d_x)$ *in* $\mathcal{E}$-**Mod**.

Proof. The sign introduced by the $\mathcal{E}$-action on the totalization and the sign coming from the various shifts $\mathcal{E}[i]$ in the construction of $\underline{A}$ cancel out. This means that the $\mathcal{E}$-action on $(M_2(R[t]), d_x)$ and $\underline{A}$ agree. The rest is clear. $\square$

We have a similar situation for the right dg-$\mathcal{E}$-module $A^\vee = \boxed{R1^\vee \xrightarrow{-x} Re_x^\vee}$ which is concentrated in degree -1 and 0. It is actually easier to work with right dg-$\mathcal{E}$-modules because the action does not get signs from shifting. Let us obtain a cofibrant replacement for $A^\vee[1]$ instead.

Note that matrices of $\mathcal{E}$ act on the right of $\mathcal{E}[i]$ simply by right multiplication. We regard elements of $A^\vee[1]$ as row vectors. For $i \geq 0$, let $M_i = \mathcal{E}[-i]$. Define $d_0^h : M_0 \to A^\vee[1]$ to be left multiplication by the matrix $\begin{pmatrix} 1 & 0 \end{pmatrix}$. For $i \geq 1$, define $d_i^h : M_i \to M_{i-1}$ to be left multiplication by the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Now we wish to totalize the chain complex

$$M = \ldots \longleftarrow 0 \longleftarrow A^\vee[1] \xleftarrow{d^h} M_0 \xleftarrow{d^h} M_1 \longleftarrow \ldots$$

in **Mod**-$\mathcal{E}$. Although we did not define totalization for right dg-modules, the definition should agree with the result we get by converting them to left dg-$\mathcal{E}^{op}$-modules. Aftering some calculation with signs, we find that the action in $\operatorname{Tot}^\oplus M$ should have no signs, unlike for totalization of left dg-modules.

Let us again consider the R-module $M_2(R[t])$. We give it the same grading as before and now let $\mathcal{E}$ act on the right by right multiplication. For a fix $x \in R$, we equip $M_2(R[t])$ with the differential

$$d_x' \begin{pmatrix} P(t) & Q(t) \\ U(t) & W(t) \end{pmatrix} = \begin{pmatrix} xQ(t) & 0 \\ \overline{P}(t) + x(P(t) - W(t)) & xQ(t) + \overline{Q}(t) \end{pmatrix}$$

Since the no signs were introduced in shifting and totalization, the $\mathcal{E}$-action of $\underline{A^\vee}[1]$ which we define to be $\operatorname{Tot}^\oplus M$ agrees with the $\mathcal{E}$-action on $M_2(R[t])$. This gives us the following.

Proposition 4.4. *We have* $\underline{A^\vee} \cong (M_2(R[t]), d_x')[-1]$ *in* **Mod**-$\mathcal{E}$.

Since $- \otimes_{\mathcal{E}} -$ is a Quillen bifunctor, the dg-R-module $\underline{A^\vee} \otimes_{\mathcal{E}} \underline{A}$ is cofibrant over R. This gives us a formula for computing $(-)_A^\wedge$ and $\operatorname{Cell}_A(-)$ when A is a length 1 Koszul complex.

Now we make some comments on general Koszul complexes. If $x_1, \ldots, x_n \in R$ is a sequence, we abbreviate the sequence to $\underline{x} = (x_1, \ldots, x_n)$. Recall that the Koszul complex of $\underline{x}$ is defined to be

$$K(\underline{x}) = K(x_1) \otimes_R \cdots \otimes_R K(x_n)$$

where $K(x_i) = \boxed{R \xrightarrow{x_i} R}$ is concentrated in degree 0 and 1.

In order to compute $A^\vee \otimes^{\mathbb{L}}_{\mathcal{E}} A$ when $A = K(\underline{x})$, we need to find cofibrant replacement for A and $A^\vee$ over $\mathcal{E}$. Since the situation with $A^\vee$ is similar to A, we will only have a discussion about A.

Let $\mathcal{E} = \mathrm{End}(K(\underline{x}))$ and $\mathcal{E}_i = \mathrm{End}_R(K(x_i))$. Since any Koszul complex is already a perfect dg-R-modules, we have

$$\mathcal{E} \cong \mathcal{E}_1 \otimes_R \cdots \otimes_R \mathcal{E}_n$$

as dg-algebras. To shorten the notation, let $\mathcal{E}' = \mathcal{E}_1 \otimes_R \cdots \otimes_R \mathcal{E}_n$. The isomorphism allows us to regard $\mathcal{E}$ as a cofibrant right dg-$\mathcal{E}'$-module, so $\mathcal{E} \otimes_{\mathcal{E}'} - : \mathcal{E}'\textbf{-Mod} \to \mathcal{E}\textbf{-Mod}$ is a Quillen equivalence.

The dg-R-module $K(\underline{x})$ has a natural dg-$\mathcal{E}'$-module structure. However, the cofibrant replacement $\underline{K(x_i)} \to K(x_i)$ and $\underline{K(x_i)^\vee} \to K(x_i)$ we gave earlier are not themselves cofibrations over $\mathcal{E}_i$. Thus, we cannot expect that the obvious $\mathcal{E}'$-linear map $\underline{K(x_1)} \otimes_R \cdots \otimes_R \underline{K(x_n)} \to K(\underline{x})$ to be a cofibrant replacement.

If we are willing to impose some conditions on R, then we can say a bit more. Recall that we have the following Künneth formula [11, 3.6.3]. For any two chain complexes $(C, d_C), (C', d_{C'})$ over R, if one of them, say C', is degreewise flat and $d_{C'}(C')$ is also degreewise flat, then there is a short exact sequence

$$\bigoplus_{p+q=n} H_p(C) \otimes_R H_q(C') \rightarrowtail H_n(C \otimes_R C') \twoheadrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C), H_q(C'))$$

Each degree of $\underline{K(x_i)}$ is already free over R, so it is obviously flat over R. Except for the image of $\underline{K(x_i)}_1 \to \underline{K(x_i)}_0$, the differential d_{x_i} of $\underline{K(x_i)}$ has flat image already. Let $J_n : R^n \to R^n$ be given by the "Jordan block" matrix

$$\begin{pmatrix} x & -1 & 0 & \dots & 0 & 0 \\ 0 & x & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x & -1 \\ 0 & 0 & 0 & \dots & 0 & x \end{pmatrix}$$

The image of $\underline{K(x_i)}_1 \to \underline{K(x_i)}_0$ is $\bigoplus_{i \geq 0} J_1(R) \oplus \varinjlim J_n(R^n)$. Therefore, we can have flatness if $J_n(R^n)$ are all flat R-modules. In such a scenario, the $\mathcal{E}'$-map $\underline{K(x_1)} \otimes_R \cdots \otimes_R \underline{K(x_n)} \to K(\underline{x})$ is a cofibrant replacement by the Künneth formula.

A Characterizations of compact dg-modules

We give a proof of proposition 3.7 by breaking parts of it into lemmas. First, we give an inductive procedure for constructing thick(R).

Let $\mathcal{R}_1$ be the strictly full subcategory of $D(R)$ containing all finite coproducts of some shifts of R. For $n \geq 1$, let $\mathcal{R}_{n+1}$ be the full subcategory of $D(R)$ consisting of all objects isomorphic to a summand of some $X \in D(R)$ which fits inside a triangle $Y \to X \to Z \to Y[1]$ with $Y \in \mathcal{R}_n$ and $Z \in \mathcal{R}_1$. Let $\mathcal{R} = \bigcup_{n \geq 1} \mathcal{R}_n$.

Lemma A.1. *We have* $\mathcal{R} = \mathrm{thick}(R)$.

Proof. As $\mathrm{thick}(R)$ is closed under all operations used to construct each $\mathcal{R}_n$, we have $\mathcal{R} \subset \mathrm{thick}(R)$. Conversely, we know that $\mathcal{R}$ is additive and closed under shifts. If we can show that $\mathcal{R}$ is closed under taking cones, then we have $\mathcal{R} = \mathrm{thick}(R)$ because $\mathcal{R}$ is clearly thick.

We prove the following claim by induction: whenever we have a triangle $X \to Z \to Y \to X[1]$ where $Z \in \mathcal{R}_n$ and $Y \in \mathcal{R}_m$, then we have $X \in \mathcal{R}_{n+m}$. The claim is obvious when $n = 1$. Now, we assume that $n \geq 2$. Since we can add two triangles, we can assume without loss of generality that Z itself fits inside a triangle of the form $Z \to V \to W \to Z[1]$, where $V \in \mathcal{R}_1$ and $W \in \mathcal{R}_{n-1}$. Complete the composite $X \to V = X \to Z \to V$ to a triangle $X \to V \to U \to X[1]$. By the octahedral axiom, we have a triangle $Y \to U \to W \to Y[1]$. By induction, we see that $U \in \mathcal{R}_{n+m-1}$, so the claim is proven. $\square$

Lemma A.2. *A compact semi-free dg-module A is inside $\mathcal{R}$.*

Proof. Let $A_1 \subset A_2 \subset \dots$ be a semi-free filtration for A. Since A is compact, applying $\mathrm{Hom}_{D(R)}(A, -)$ to the triangle $\bigoplus A_i \to \bigoplus A_i \to \varinjlim A_i \to (\bigoplus A_i)[1]$ and using the five-lemma shows that $\mathrm{Hom}_{D(R)}(A, A) = \varinjlim \mathrm{Hom}_{D(R)}(A, A_i)$. In particular, this means that the identity of A factors through some A_n.

We now show that $A \in \mathcal{R}_n$ by induction on n. Let $\mathcal{F}$ be the set of all finite coproduct of shifts of R. If $n = 1$, then applying $\mathrm{Hom}_{D(R)}(A, -)$ to A_1 and using compactness shows that A is a retract of some dg-module in $\mathcal{F}$. Thus, we have $A \in \mathcal{R}_1$.

Suppose now that $n \geq 2$. The composite $A \to A_n \to A_n/A_{n-1}$ factors through some $F \in \mathcal{F}$. We may complete the morphism $A \to F$ to a triangle to obtain the following commutative diagram

$$\begin{array}{ccccccc}
A' & \longrightarrow & A & \longrightarrow & F & \longrightarrow & A'[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A_{n-1} & \longrightarrow & A_n & \longrightarrow & A_n/A_{n-1} & \longrightarrow & A_{n-1}[1]
\end{array}$$

where the rows are triangles. The object A' is compact because A and F are. Thus, by induction, the map $A' \to A_{n-1}$ factors through some $G \in \mathcal{R}_{n-1}$.

Consider the composite $F \to G[1] = F \to A'[1] \to G[1]$ and complete it to a triangle $G \to H \to F \to G[1]$. Then by definition $H \in \mathcal{R}_n$. We have the following commutative diagram

$$
\begin{array}{ccccccc}
A' & \longrightarrow & A & \longrightarrow & F & \longrightarrow & A'[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
G & \longrightarrow & H & \longrightarrow & F & \longrightarrow & G[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A_{n-1} & \longrightarrow & A_n & \longrightarrow & A_n/A_{n-1} & \longrightarrow & A_{n-1}[1]
\end{array}
$$

The composite $A \to H \to A_n$ may not be the same as the map $A \to A_n$ we had at the beginning. Therefore, we take their difference and call it $h : A \to A_n$. Then we can lift h to a morphism $\widetilde{h} : A \to A_{n-1}$ by the following commutative diagram

$$
\begin{array}{ccccccc}
A & = & A & \longrightarrow & 0 & \longrightarrow & A[1] \\
\downarrow{\scriptstyle \widetilde{h}} & & \downarrow{\scriptstyle h} & & \downarrow & & \downarrow \\
A_{n-1} & \longrightarrow & A_n & \longrightarrow & A_n/A_{n-1} & \longrightarrow & A_n[1]
\end{array}
$$

By induction again, we can factor $\widetilde{h}$ through some $G' \in \mathcal{R}_{n-1}$. Then the original map $A \to A_n$ factors through $G' \oplus F \in \mathcal{R}_n$. Since A is a retract of A_n, it is a retract of $G' \oplus F$, so $A \in \mathcal{R}_n$. □

Let $\mathcal{P}$ be the full subcategory of dg-modules satisfying (3) of proposition 3.7 in $D(R)$. We now prove proposition 3.7. Although the proof can be given in purely triangulated language by proposition 3.6, we will use the projective model structure of R-**Mod** as a shortcut in places.

Proof of 3.7. We first prove that (2) and (3) are equivalent by showing $\mathcal{P} = \mathcal{R}$. The containment $\mathcal{P} \subset \mathcal{R}$ follows directly from the definition of $\mathcal{R}$. For the converse, we show by induction that $\mathcal{R}_n \subset \mathcal{P}$.

The case $n = 1$ is obvious. Assume that $A \in \mathcal{R}_n$ where $n \geq 2$. Then A fits inside a triangle $A_{n-1} \to A \to A_1 \to A_{n-1}[1]$ with $A_1 \in \mathcal{R}_1$ and $A_{n-1} \in \mathcal{R}_{n-1}$. By induction and adding two obvious triangles to $A_{n-1} \to A \to A_1 \to A_{n-1}[1]$, we may assume that A_{n-1} admits a filtration like in (3) and A_1 is a finite coproduct of shifts of R.

Let the map $f : A_1[-1] \to A_{n-1}$ be obtained by rotating the triangle. Since $A_1[-1]$ is cofibrant and A_{n-1} is fibrant, the map f is actually represented by a map $A_1[-1] \to A_{n-1}$ in R-**Mod**, which we also call f. Then we have a commutative diagram

$$
\begin{array}{ccccccc}
A_{n-1} & \longrightarrow & A & \longrightarrow & A_1 & \longrightarrow & A_{n-1}[1] \\
\| & & \downarrow{\scriptstyle \wr} & & \| & & \| \\
A_{n-1} & \longrightarrow & \mathrm{cone}(f) & \longrightarrow & A_1 & \longrightarrow & A_{n-1}[1]
\end{array}
$$

in $D(R)$. This proves that $A \in \mathcal{R}_n$.

We now prove that (1) is equivalent to (2) by showing $D(R)^c = \text{thick}(R)$. Let $A \in D(R)^c$. We can assume that A is cofibrant by taking a cofibrant replacement of A. Since $\mathcal{R}$ is closed under retract, by the characterization 2.3, we can assume that A is semi-free. By lemma A.2 and lemma A.1, we have $A \in \text{thick}(R)$.

Showing that $D(R)^c$ contains $\text{thick}(R)$ amounts to showing that direct summands of compact objects are compact. Suppose that $A, A' \in D(R)$ are such that $A \oplus A'$ is compact. We have the triangles $A \to A \oplus A' \to A' \to A[1]$ and $A' \to A \oplus A \to A \to A'[1]$. Take a set of objects $X_i \in D(R)$. Then applying $\text{Hom}_{D(R)}(-, \bigoplus X_i)$ to the second triangle and gives the commutative diagram

$$\begin{array}{ccccc} \bigoplus \text{Hom}_{D(R)}(A, X_i) & \rightarrowtail & \bigoplus \text{Hom}_{D(R)}(A \oplus A', X_i) & \twoheadrightarrow & \bigoplus \text{Hom}_{D(R)}(A', X_i) \\ \downarrow & & \downarrow \wr & & \downarrow \\ \text{Hom}_{D(R)}(A, \bigoplus X_i) & \rightarrowtail & \text{Hom}_{D(R)}(A \oplus A', \bigoplus X_i) & \twoheadrightarrow & \text{Hom}_{D(R)}(A', \bigoplus X_i) \end{array}$$

with exact rows. Thus, the left vertical arrow is injective and the right vertical arrow is surjective. Using the first triangle shows that they are both isomorphisms, so summands of compact objects are compact.

It remains to show that (1) and (4) are equivalent. To see that (1) implies (4), note that $A^\vee \otimes_R^{\mathbb{L}} -$ and $\mathbb{R}\text{Hom}_R(A, -)$ are triangulated functors, so the full subcategory L on which they agree is a triangulated subcategory of $D(R)$. Since A is compact, we see that $\mathbb{R}\text{Hom}_R(A, -)$ commutes with all coproducts. This means that L is a localizing subcategory of $D(R)$. By definition, we have $R \in L$, so $L = D(R)$.

Finally, (4) implies (1) because $A^\vee \otimes_R^{\mathbb{L}} -$ commutes with coproducts. □

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