CONJUGACY CLASSES OF FULL SUBGROUPS OF $N_{\mathrm{Sp}(2)}(T)$

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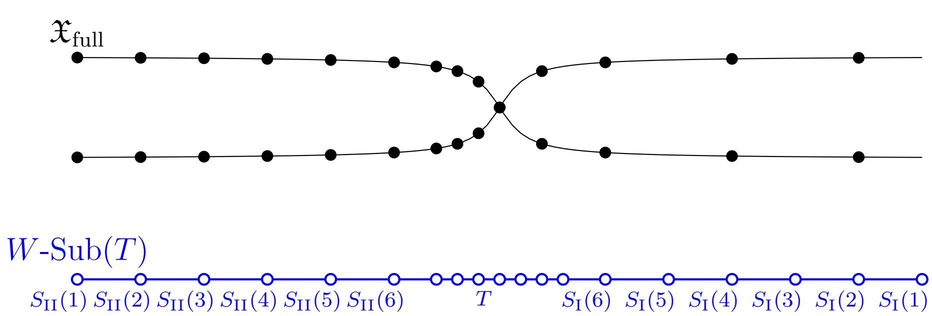
0. Introduction

Spaces of conjugacy classes of subgroups occur very naturally in the study of equivariant cohomology theories. This note is concerned with explicit computations for compact Lie groups. We give neighborhood bases for the block of full subgroups $\mathfrak{X}_{\text{full}}$ in the space of conjugacy classes of $G = N_{\mathrm{Sp}(2)}(T)$. The findings are summarized by the following picture.



The space $\mathfrak{X}_{\text{full}}$ can be thought of as a "ramified double cover" of the space W-Sub(T) of W-invariant subgroups of T. The curves and lines in the picture are only meant to suggest a continuous imagery; they are not part of the spaces. The conjugacy class $[G]$ is the only limit point in $\mathfrak{X}_{\text{full}}$ and every other point is isolated.

In section 1, we introduce some notations and review a couple of concepts that are possibly familiar to the readers. Then we identify the set underlying $\mathfrak{X}_{\text{full}}$ in section 2, which is probably the most elaborate part of the note. We take a short detour to compute the Weyl groups associated to the subgroups of G in section 3. Finally, we discuss the topology of $\mathfrak{X}_{\text{full}}$ in section 4.

We adopt the method in [4] to identify $\mathfrak{X}_{\text{full}}$. We go through some of the arguments there in the present note to be relatively self-contained.

1. Notation

It is well-known that the Weyl group of Sp(2) is

$$W = D_8 = \langle r, s | r^4 = s^2 = 1 \rangle.$$

We denote the maximal torus of Sp(2) by T. Let the normalizer of T in Sp(2) be G, then we have the short exact sequence

$$1 \longrightarrow T \hookrightarrow G \xrightarrow{\pi} W \longrightarrow 1.$$

We call a subgroup $H \subset G$ **full** if $\pi(H) = W$.

1.1. The group Sp(2).

The division ring of quaternions is denoted $\mathbb{H}$. The vector space $\mathbb{H}^2$ over $\mathbb{H}$ has the standard symplectic form given by the formula

$$\left\langle \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right\rangle = h_1\overline{k_1} + h_2\overline{k_2}.$$

where $h_1, h_2, k_1, k_2 \in \mathbb{H}$ and $\overline{k_1}$ means the quaternion conjugate of k_1. The group Sp(2) is the subgroup of invertible $\mathbb{H}$-linear maps $\mathbb{H}^2 \to \mathbb{H}^2$ that preserve the standard symplectic form.

If we restrict the scalar multiplication on $\mathbb{H}^2$ to $\mathbb{C}$, we may identify $\mathbb{H}^2$ with the $\mathbb{C}$-vector space $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2 j$. The identification we are making is explicitly given by

$$\begin{pmatrix} a_1 + b_1 i + c_1 j + d_1 k \\ a_2 + b_2 i + c_2 j + d_2 k \end{pmatrix} \in \mathbb{H}^2 \quad \leftrightarrow \quad \begin{pmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ c_1 + d_1 i \\ c_2 + d_2 i \end{pmatrix} \in \mathbb{C}^4.$$

As the standard symplectic form on $\mathbb{H}^2$ correspond to the standard Hermitian form for $\mathbb{C}^4$ under the above identification, the maps in Sp(2) can be now be written as 4×4 unitary matrices. From now on, by Sp(2), we shall mean its image under the embedding into U(4).

The maximal torus of Sp(2) is simply

$$T = \left\{ \begin{pmatrix} z & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & \overline{z} & 0 \\ 0 & 0 & 0 & \overline{w} \end{pmatrix} : z, w \in \mathbb{C}, |z| = |w| = 1 \right\}$$

1.2. **W-modules.** We choose a section $\sigma : W \to G$ of π by letting

$$\sigma(r) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \sigma(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and letting $\sigma(r^i s^j) = \sigma(r)^i \sigma(s)^j$ for $0 \le i \le 3, 0 \le j \le 1$. The section σ gives rise to a right action of W on T defined as $t^w = \sigma(w)^{-1} t \sigma(w)$. This makes T into a right W-module.

Remark 1. The W-action on T is typically defined by observing that the conjugation action of G on T descends to an action of W. In other words, it does not matter how we lift elements of W to G to act on T. In particular, we may pick any other section τ and we would have $\tau(w)^{-1} t \tau(w) = \sigma(w)^{-1} t \sigma(w)$. This will be used later.

We chose a specific section σ only for concreteness. If one unravels the proof that the Weyl group of $\mathrm{Sp}(2)$ is D_8, the section σ should be the obvious one to pick. The reader may forget about the section σ if he wishes.

Given a full subgroup $H \subset G$, we can fit it into a commutative diagram

$$(*) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & S & \hookrightarrow & H & \longrightarrow & W & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & T & \hookrightarrow & G & \xrightarrow{\pi} & W & \longrightarrow & 1 \end{array}$$

where S is a subgroup of T and both rows are exact. The commutative diagram also indicates that the W-action on T restricts to a W-action on S. This means that we only have to care about the W-invariant subgroups of T. Moreover, for each full subgroup H of G, we can pick some section $\tau : W \to H$ of $\pi : H \to W$, which is also a section of $\pi : G \to W$ under the subgroup inclusion $H \hookrightarrow G$. Thus, we may write H in the standard form $H = H(S, \tau) = \{s\tau(w) : s \in S, w \in W\}$.

The problem of classifying conjugacy classes of full subgroups of G reduces to the following questions:

(1) For which W-submodule S of T is there a subgroup $H(S, \tau)$?
(2) If $H(S, \tau)$ does exist for a particular S, how many conjugacy classes are there for this S?

Group cohomology gives a good framework to think about and answer these questions. We now bring in Pontrjagin duality because it is a nice way to organize the W-submodules of T. This also prepares us for the group cohomology calculations later.

We will work with T additively, so we use the identification $T \cong (\mathbb{R}/\mathbb{Z})^2$ from now on. Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle group. For each right W-submodule $S \subset T$, we can give the Pontrjagin dual $S^* = \operatorname{Hom}(S, S^1)$ a left W-module structure by the action $wf(t) = f(t^w)$ for each $f \in S^*, w \in W$. Let $\Lambda^S = \ker(T^* \to S^*)$ where the map $T^* \to S^*$ is induced by the inclusion $S \hookrightarrow T$.

As an example, take the trivial submodule 0 of T, then $\Lambda^0 = T^*$ can be identified with $\mathbb{Z}^2$, where elements of $\mathbb{Z}^2$ are considered as column vectors. The group $\mathrm{GL}_2(\mathbb{Z})$ has a left action on $\mathbb{Z}^2$ by multiplication. The W-module structure on Λ^0 is then given by the map $\rho : W \to \mathrm{GL}_2(\mathbb{Z})$ that sends

$$r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proposition 1.1. *The W-submodules of Λ^0 are given below.*

(1) *The lattice* $\Lambda^S_{\mathrm{I}}(m) = \left\langle \begin{pmatrix} m \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ m \end{pmatrix} \right\rangle, m \geq 1.$

(2) *The lattice* $\Lambda^S_{\mathrm{II}}(m) = \left\langle \begin{pmatrix} m \\ -m \end{pmatrix}, \begin{pmatrix} m \\ m \end{pmatrix} \right\rangle, m \geq 1.$

(3) *The trivial W-submodule 0.*

Proof. The proof is completely routine. Any subgroup of Λ^0 is free abelian of rank at most 2. The rank 0 W-submodule is obviously trivial and there are no rank 1 submodules because it has to be stable under left action by $r \in W$.

Note that $\Lambda^S_{\mathrm{I}}(m)$ and $\Lambda^S_{\mathrm{II}}(m)$ are W-submodules of Λ^0. We show that any W-submodule $\Lambda \subset \Lambda^0$ is either $\Lambda^S_{\mathrm{I}}(m)$ or $\Lambda^S_{\mathrm{II}}(m)$. There is a vector $(p, q) \neq 0$ in Λ with the minimum length. We claim that (p, q) together with $r(p, q) = (-q, p)$ forms a basis for Λ.

They are linearly independent over $\mathbb{Z}$ because they are linearly independent over $\mathbb{R}$ by checking a determinant. Any vector $(k, l) \in \Lambda^0 \setminus \Lambda$ must be within $\sqrt{2(p^2 + q^2)}/2$ of a point in $\langle (p, q), (-q, p) \rangle$, so $\Lambda = \langle (p, q), (-q, p) \rangle$ by the minimality of (p, q).

Now we know $(p, -q) = s(p, q) \in \langle (p, q), (-q, p) \rangle$. By solving an appropriate 2×2 linear system, we find that $(p^2 - q^2)/(p^2 + q^2) \in \mathbb{Z}$. This means $p = 0$, $q = 0$ or $|p| = |q|$, ending the proof. □

2. Group cohomology calculations

We first review some basic facts from group cohomology. The material is based on [4, §3]. Using the lattice dual functor and a Künneth formula, we compute the relevant cohomology groups and describe the set underlying $\mathfrak{X}_{\text{full}}$.

2.1. **Low-dimensional cohomology.** If M is a W-module, we let $C^*(W;M)$ be the cochain complex obtained by applying $\mathrm{Hom}_W(-,M)$ to the bar resolution of $\mathbb{Z}$ over $\mathbb{Z}W$. When we speak of cocycles, coboundaries and differentials, these will come from $C^*(W;M)$.

Recall that for a section $\tau : W \to G$, the factor set of τ is the function $f_\tau : W \times W \to T$ defined by the formula $f_\tau(v,w) = \tau(vw)^{-1}\tau(v)\tau(w)$ for each pair $(v,w) \in W \times W$. The factor set f_τ is a 2-cocycle representing the extension class $\epsilon(G)$ in $H^2(W;T)$.

Here is how one can classify conjugacy classes of the full subgroups of G. For each W-submodule of S, we have a short exact sequence

$$0 \longrightarrow S \hookrightarrow T \longrightarrow T/S \longrightarrow 0.$$

This induces a long exact sequence in group cohomology. The part that interests us is the following five-term exact sequence

$$H^1(W;T) \to H^1(W;T/S) \to H^2(W;S) \to H^2(W;T) \to H^2(W;T/S).$$

Proposition 2.1 ([4], lemma 3.3)**.** *For each S, there is a section $\tau : W \to G$ giving a full subgroup $H(S,\tau)$ fitting into the diagram*

$$(*)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & S & \hookrightarrow & H & \longrightarrow & W & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & T & \hookrightarrow & G & \xrightarrow{\pi} & W & \longrightarrow & 1 \end{array}$$

if and only if $\epsilon(G)$ lifts to $H^2(W;S)$.

We make a simple calculation first.

Lemma 2.2 ([4], lemma 3.2)**.** *Take a section $\tau : W \to G$ and a function $g : W \to T$. Then $f_{\tau g} = f_\tau \delta g$.*

The section τg is defined by pointwise multiplication. The δ is the differential in the cochain complex obtained from the bar resolution. We could also write the formula in additive notation since the values are in T. In additive notation, $f_{\tau g} = f_\tau + \delta g$.

Proof. Take $v, w \in W$. Note that by definition $\delta g(v,w) = g(v)^w g(vw)^{-1} g(w)$. We calculate

$$\begin{aligned} f_{\tau g}(v,w) &= g(vw)^{-1}\tau(vw)^{-1}\tau(v)g(v)\tau(w)g(w) \\ &= g(vw)^{-1}\tau(vw)^{-1}\tau(v)\tau(w)g(v)^w g(w) \\ &= g(vw)^{-1} f_\tau(v,w) g(v)^w g(w). \end{aligned}$$

As everything in the last line is in T, which means they commute, we have $f_{\tau g}(v,w) = f_\tau(v,w)\delta g(v,w)$ as needed. □

Proof of 2.1. If there is a section τ giving a full subgroup $H(S,\tau)$, then the factor set $f_\tau : W \times W \to T$ actually takes values in S, so $\epsilon(G)$ lifts.

Conversely, if $\epsilon(G)$ in $H^2(W;T)$ lifts to $H^2(W;S)$, there is a 2-cocycle $z : W \times W \to S$ representing the lift. Then by the inclusion of S into T, we can think of z as a 2-cocycle representing $\epsilon(G)$ in $H^2(W;T)$. We choose any section τ so that f_τ is a 2-cocycle representing $\epsilon(G)$.

The 2-cocycle $z - f_\tau$ is a 2-coboundary δg. We see that $z = f_\tau + \delta g = f_{\tau g}$, so we have a subgroup $H(S, \tau g)$ fitting into the diagram $(*)$ as needed. ☐

Proposition 2.3 ([4], lemma 3.3)**.** *Suppose there is a section $\tau : W \to G$ giving a subgroup $H(S,\tau)$ for a given S, the number of conjugacy classes for S is exactly the cardinality of $H^1(W;T/S)$.*

Again, we need a couple of calculational lemmas first.

Lemma 2.4. *Two full subgroups $H(S,\tau)$ and $H(S,\tau')$ are G-conjugate if and only if they are T-conjugate.*

Proof. One way is obvious. Suppose that $g^{-1}H(S,\tau)g = H(S,\tau')$. As τ is a section, we can write $g = \tau(w)t$ for some $w \in W$ and $t \in T$. Note the order reversal from the typical order that we used. Then $t^{-1}H(S,\tau)t = t^{-1}\tau(w)^{-1}H(S,\tau)\tau(w)t = g^{-1}H(S,\tau)g = H(S,\tau')$. ☐

Lemma 2.5 ([4], lemma 3.2)**.** *For any $t \in T$, $tH(S,\tau)t^{-1} = H(S,\tau\delta t)$.*

To be clear, the section $\tau\delta t$ is obtained by pointwise multiplication of τ with the 1-coboundary δt.

Proof. Let $s \in S$ and $w \in W$. Note that by definition, $\delta t(w) = \tau(w)^{-1}t\tau(w)t^{-1}$. The lemma follows from the calculation

$$ts\tau(w)t^{-1} = st\tau(w)t^{-1} = s\tau(w)\tau(w)^{-1}t\tau(w)t^{-1} = s\tau(w)\delta t(w).$$

☐

Lemma 2.6 ([4], lemma 4.1)**.** *If $g : W \to T$ is a function, then $H(S,\tau) = H(S,\tau g)$ if and only if the values of g are actually in S.*

This is self-evident. The collection of all full subgroups of G is denoted by $\mathrm{Sub}(G)_{\mathrm{full}}$. The notation will be reintroduced in section 4.

Proof of 2.3. Let us fix some section τ_0 so that we have a subgroup $H(S,\tau_0)$. We have an onto map $\kappa : C^1(W;T) \to \mathrm{Sub}(G)_{\mathrm{full}}$ that sends g to $H(S,\tau_0 g)$. By lemma 2.6, κ induces a bijection $C^1(W;T/S) \to \mathrm{Sub}(G)_{\mathrm{full}}$. Then by lemma 2.4 and lemma 2.5, κ induces a bijection $H^1(W;T/S) \to \mathrm{Sub}(G)_{\mathrm{full}}/G$. ☐

2.2. **Lattice dual.** The Pontrjagin duality functor converts questions about mod-$W(T)$ to questions about W-mod(Λ^0). The latter category is nicer because we only need to do linear algebra. However, Pontrjagin duality is contravariant, so we cannot use it in group cohomology calculations. To get a covariant functor, we use the lattice dual functor $(-)^\vee = \mathrm{Hom}(-, \mathbb{Z})$.

For each W-submodule Λ^S of Λ^0, we define $\Lambda_S = (\Lambda^S)^\vee$. From the left W-module structure of Λ^S, we obtain a right W-module structure on Λ_S by setting $\theta^w(f) = \theta(wf)$ for each $\theta \in \Lambda_S$ and each $w \in W$. The W-module structure of Λ_S can be described as follows.

Proposition 2.7. *As W-modules, $\Lambda_S \cong \Lambda^S$.*

Proof. If $\Lambda^S = 0$, there is nothing to prove. With that out of the way, there are two cases to deal with. First, consider the W-module $\Lambda^S_{\mathrm{I}}(m)$ and its dual $\Lambda_{S,\mathrm{I}}(m) = \Lambda^S_{\mathrm{I}}(m)^\vee$. Consider the vectors

$$e_1 = \begin{pmatrix} m \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ m \end{pmatrix}, \quad e_1^\vee = \begin{pmatrix} \frac{1}{m} & 0 \end{pmatrix}, \quad e_2^\vee = \begin{pmatrix} 0 & \frac{1}{m} \end{pmatrix}.$$

The vectors e_1, e_2 form an ordered basis for the abelian group $\Lambda^S_{\mathrm{I}}(m)$. Similarly, $e_1^\vee, e_2^\vee$ forms an ordered basis for $\Lambda_{S,\mathrm{I}}(m)$. We have an isomorphism $\phi : \Lambda^S_{\mathrm{I}}(m) \to \Lambda_{S,\mathrm{I}}(m)$ of abelian groups by sending e_1, e_2 to $e_1^\vee, e_2^\vee$ respectively. Then one can check that ϕ is in fact W-equivariant, so ϕ is an isomorphism of W-modules.

For the W-modules $\Lambda^S_{\mathrm{II}}(m)$ and $\Lambda_{S,\mathrm{II}}(m)$, we have the ordered basis $\{f_1, f_2\}$ and $\{f_1^\vee, f_2^\vee\}$ respectively, where

$$f_1 = \begin{pmatrix} m \\ -m \end{pmatrix}, \quad f_2 = \begin{pmatrix} m \\ m \end{pmatrix}, \quad f_1^\vee = \begin{pmatrix} \frac{1}{2m} & -\frac{1}{2m} \end{pmatrix}, \quad f_2^\vee = \begin{pmatrix} \frac{1}{2m} & \frac{1}{2m} \end{pmatrix}.$$

Similarly to before, we can check that the map $\psi : \Lambda^S_{\mathrm{II}}(m) \to \Lambda_{S,\mathrm{II}}(m)$ sending f_1, f_2 to $f_1^\vee, f_2^\vee$ respectively is a W-module isomorphism. ☐

A perhaps subtle point is that although $\Lambda^S_{\mathrm{I}}(m)$ and $\Lambda^S_{\mathrm{II}}(m)$ are isomorphic as abelian groups, they are not isomorphic as W-modules. They are, however, related by restriction of scalars.

Consider the automorphism of D_8 defined by $\alpha(r) = r$, $\alpha(s) = r^3 s$. This gives an automorphism α of $\mathbb{Z}D_8$. This automorphism gives a functor $\mathrm{Res} : W\text{-}\mathbf{mod} \to W\text{-}\mathbf{mod}$ sending a W-module M to the W-module M' where M' is the same as M as an abelian group, but $w \cdot m = \alpha(w)m$ for each $m \in M'$.

Proposition 2.8. *We have $\mathrm{Res}\,\Lambda^S_{\mathrm{II}}(m) \cong \Lambda^S_{\mathrm{I}}(m)$ as W-modules.*

Proof. We continue using the notation from 2.7. Write the scalar multiplication in $\operatorname{Res}\Lambda^S_{\mathrm{II}}(m)$ as $\cdot$, we have

$$r\cdot f_1 = rf_1 = f_2,\quad r\cdot f_2 = rf_2 = -f_1$$

and

$$s\cdot f_1 = r^3 s f_1 = f_1,\quad s\cdot f_2 = r^3 s f_2 = -f_2.$$

Thus, we have a W-module isomorphism $\operatorname{Res}\Lambda^S_{\mathrm{II}}(m)\cong \Lambda^S_{\mathrm{I}}(m)$ by sending f_1, f_2 to e_1, e_2 respectively. ☐

Let us relate this new construction to the five-term exact sequence we gave earlier. We have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda_0 & \hookrightarrow & \mathbb{R}^2 & \longrightarrow & T & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Lambda_S & \hookrightarrow & \mathbb{R}^2 & \longrightarrow & T/S & \longrightarrow & 0 \end{array}$$

As multiplying by $|W| = 8$ is an isomorphism of $\mathbb{R}\to\mathbb{R}$, by using the transfer map [2, p. 83], we obtain $H^i(W;\mathbb{R}) = 0$ for $i \geq 1$. Looking at the long exact sequence for the two rows and applying naturality, we obtain the commutative squares

$$\begin{array}{ccc} H^i(W;T) & \longrightarrow & H^i(W;T/S) \\ \downarrow\wr & & \downarrow\wr \\ H^{i+1}(W;\Lambda_0) & \longrightarrow & H^{i+1}(W;\Lambda_S) \end{array}$$

Now, the five-term exact sequence looks like

$$\begin{array}{ccccccccc} H^1(W;T) & \longrightarrow & H^1(W;T/S) & \to & H^2(W;S) & \longrightarrow & H^2(W;T) & \longrightarrow & H^2(W;T/S) \\ \downarrow\wr & & \downarrow\wr & & & & \downarrow\wr & & \downarrow\wr \\ H^2(W;\Lambda_0) & \longrightarrow & H^2(W;\Lambda_S) & & & & H^3(W;\Lambda_0) & \longrightarrow & H^3(W;\Lambda_S). \end{array}$$

The main tool of calculation is the following Künneth formula [5].

Proposition 2.9. *Let G, G' be finite groups and M, M' be a G-module and a G'-module respectively. If M, M' are, as modules over $\mathbb{Z}$, both finitely generated and free, then we have a short exact sequence*

$$0\to \bigoplus_{p+q=n} H^p(G;M)\otimes H^q(G';M') \to H^n(G\times G'; M\otimes M')$$
$$\to \bigoplus_{p+q=n+1} \operatorname{Tor}_1^{\mathbb{Z}}(H^p(G;M), H^q(G';M')) \to 0.$$

Proposition 2.10. *For $n \geq 0$, we have*

$$H^n(W;\Lambda_{\mathrm{I}}^S(m)) = \begin{cases} (\mathbb{Z}/2)^i & \text{if } n = 2i; \\ (\mathbb{Z}/2)^{i+1} & \text{if } n = 2i+1. \end{cases}$$

Moreover, $H^n(W;\Lambda_{\mathrm{II}}^S(m))) \cong H^n(W;\Lambda_{\mathrm{I}}^S(m))$ as abelian groups.

Proof. Since restriction of scalars is exact, we obtain the isomorphism of abelian groups $H^n(W;\Lambda_{\mathrm{II}}^S(m))) \cong H^n(W;\Lambda_{\mathrm{I}}^S(m))$.

Now we compute $H^n(W;\Lambda_{\mathrm{I}}^S(m))$ by building it up from simpler pieces. It is well-known that for $p,q \geq 0$, we have

$$H^p(C_2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & p \neq 0 \text{ odd} \\ \mathbb{Z}/2 & p \neq 0 \text{ even} \end{cases} \qquad H^p(C_2;\widetilde{\mathbb{Z}}) = \begin{cases} 0 & p \text{ even} \\ \mathbb{Z}/2 & p \text{ odd} \end{cases}$$

where $\widetilde{\mathbb{Z}}$ is the unique nontrivial C_2-module with the underlying abelian group being $\mathbb{Z}$. Using the Künneth formula given above, we find that

$$H^n(C_2 \times C_2;\mathbb{Z} \otimes \widetilde{\mathbb{Z}}) = \begin{cases} (\mathbb{Z}/2)^i & \text{if } n = 2i; \\ (\mathbb{Z}/2)^{i+1} & \text{if } n = 2i+1. \end{cases}$$

Finally, observe that $\Lambda_{\mathrm{I}}^S(m) \cong \mathrm{Coind}_{C_2 \times C_2}^W \mathbb{Z} \otimes \widetilde{\mathbb{Z}}$ as W-modules. The proposition now follows from Shapiro's lemma. $\square$

Let $A = \{(0,0), (\frac{1}{2}, \frac{1}{2})\}$ be the unique W-submodule of order 2 in T.

Proposition 2.11. *There is a subgroup H fitting into the commutative diagram*

$$(*) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & S & \hookrightarrow & H & \longrightarrow & W & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & T & \hookrightarrow & G & \xrightarrow{\pi} & W & \longrightarrow & 1 \end{array}$$

if and only if $A \subset S$.

Proof. Consider the section $\tau : W \to G$ defined by setting

$$\tau(r) = \begin{pmatrix} 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad \tau(s) = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and $\tau(r^i s^j) = \tau(r)^i \tau(s)^j$ for $0 \leq i \leq 3$ and $0 \leq i \leq 1$. We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \hookrightarrow & K & \longrightarrow & W & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & T & \hookrightarrow & G & \xrightarrow{\pi} & W & \longrightarrow & 1 \end{array}$$

where K is the subgroup of order 16 generated by $\tau(r)$ and $\tau(s)$ in G.

By explicit computation, the factor set f_τ takes values in A for $v, w \in W$. Moreover, f_τ attains all values of A. As G is not a split extension of W by T, the proposition is established. $\square$

Corollary. *The conjugacy classes of full subgroups of G are given by the following table. #Conj is short for the number of conjugacy classes.*

Lattice	*0*	$\Lambda^S_{\mathrm{I}}(m)$, *m odd*	$\Lambda^S_{\mathrm{I}}(m)$, *m even*	$\Lambda^S_{\mathrm{II}}(m)$
#Conj	*1*	*0*	*2*	*2*

3. Weyl groups of subgroups

We compute the Weyl group of full subgroups of G in this section using linear algebra. The plan is to prove a formula for the Weyl group first and then proceed to examine a bit more closely the Pontrjagin duality functor to exploit the formula. The content is based on [4, §4], but we give more or less complete details here.

The Weyl group $W_G(H)$ of a subgroup $H \subset G$ is defined as the group $N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G.[1]

For each subgroup $S \subset T$, let $S^+ = \{t \in T : t^w \in S \text{ for all } w \in W\}$. Then S^+ is also a subgroup of T and we shall denote Λ^{S^+} by Λ^S_+. We first describe the normalizer in terms of S^+.

Proposition 3.1 ([4], lemma 4.1)**.** *For a full subgroup $H = H(S, \tau)$, we have the formula $N_G(H) = H(S^+, \tau)$.*

Proof. First, we do have a subgroup $H(S^+, \tau)$ because the factor set f_τ takes values in $S \subset S^+$ by assumption.

Any $g \in G$ can be written as $g = \tau(w)t$ for some $w \in W$ and $t \in T$. Then by lemma 2.5 and lemma 2.6, $g^{-1}H(S, \tau)g \subset H(S, \tau)$ is equivalent to saying δt takes value in S. In other words, $g \in N_G(H)$ if and only if $t \in S^+$, so $N_G(H) = H(S^+, \tau)$. $\square$

Now we give an easy formula for determining $W_G(H)$.

[1] Our terminology is standard in equivariant homotopy theory.

Proposition 3.2 ([4], p. 10)**.** *For a full subgroup $H = H(S,\tau)$, we have the formula $W_G(H) \cong S^+/S \cong (\Lambda^S/\Lambda^S_+)^*$.*

Proof. Consider the composition $S^+ \hookrightarrow H(S^+,\tau) = N_G(H) \to N_G(H)/H$. If $t\tau(w)$ is in the kernel of the composite with $t \in S^+$ and $w \in W$, then $t\tau(w) = s\tau(v)$ for some $s \in S$ and $v \in W$. Applying π to both sides shows $v = w$ and so $t = s \in S$. Thus, $W_G(H) \cong S^+/S$.

The isomorphism $S^+/S \cong \Lambda^S/\Lambda^S_+$ comes from Pontrjagin duality. Since Pontrjagin duality is exact, by applying $(-)^*$ to the short exact sequence

$$0 \longrightarrow S \hookrightarrow T \longrightarrow T/S \longrightarrow 0.$$

we get the short exact sequence

$$0 \longrightarrow \Lambda^S \hookrightarrow T^* \longrightarrow S^* \longrightarrow 0.$$

Similarly, we have $(T/S^+)^* = \Lambda^S_+$. Applying the Pontrjagin duality functor once more to the short exact sequence

$$0 \longrightarrow S \hookrightarrow S^+ \longrightarrow S^+/S \longrightarrow 0$$

and using one of the isomorphism theorems, we get $(S^+/S)^* \cong \Lambda^S/\Lambda^S_+$. By the double duality isomorphism, we have $S^+/S^* \cong (\Lambda^S/\Lambda^S_+)^*$. □

As the Pontrjagin dual of a finite abelian group is itself, when the index $[\Lambda^S : \Lambda^S_+]$ is finite, we have the formula $W_G(H) \cong \Lambda^S/\Lambda^S_+$.

3.1. **Translating between lattices and subgroups.** There are two ways to calculate the Weyl groups from proposition 3.2. As we have been working with lattices via Pontrjagin duality so far, we first need a standard fact about Pontrjagin duality to help us translate between lattices and the subgroups of T. Then we describe how to calculate S^+ and Λ^S_+ from S and Λ^S respectively, so we may compute the Weyl groups either way.

Proposition 3.3. *Given a (closed) subgroup S of T and a $t \in T \setminus S$, there is some $f \in S^*$ such that $f(s) = 0$ for every $s \in S$ but $f(t) \neq 0$.*

A proof of proposition 3.3 is given in [7, p. 75].

Under the double duality isomorphism, we can identify $s \in S$ with a map $s^{**} : \Lambda^0 \to S^1$ sending $f \in \Lambda^0$ to $f(s) \in S^1$.

Proposition 3.4. *We have $S = \ker((\Lambda^0)^* \to (\Lambda^S)^*)$ for subgroups S of T.*

Proof. The inclusion $S \subset \ker((\Lambda^0)^* \to (\Lambda^S)^*)$ follows from the definition of Λ^0 and Λ^S. The reverse containment is given by proposition 3.3. □

Here is a more conceptual way of thinking about the situation. Let **mod-**$W(T)$ and W**-mod**(Λ^0) be the category of right W-submodules of T and the category of left W-submodules of Λ^0 respetively. The contravariant functor $\Lambda^- : \textbf{mod-}W(T) \to W\textbf{-mod}(\Lambda^0)$ sending $S \to \Lambda^S$ is an anti-isomorphism of categories by proposition 3.4.

We now give an explicit description of **mod-**$W(T)$ like we have done for W**-mod**(Λ^0) in proposition 1.1.

Proposition 3.5. *The W-submodules of T are given below.*

(1) The W-submodule

$$S_{\mathrm{I}}(m) = \left\{ \left(\frac{k}{m}, \frac{l}{m} \right) : 0 \le k < m, 0 \le l < m \right\}$$

for $\Lambda^S_{\mathrm{I}}(m)$.

(2) The W-submodule

$$S_{\mathrm{II}}(m) = \left\{ \left(\frac{k}{2m}, \frac{l}{m} + \frac{k}{2m} \right) : 0 \le k < 2m, 0 \le l < m \right\}$$

for $\Lambda^S_{\mathrm{II}}(m)$.

(3) The W-module T for 0.

Proof. We prove the claim for $\Lambda^S_{\mathrm{I}}(m)$. The proof for $\Lambda^S_{\mathrm{II}}(m)$ is similar. Since $S \cong (\Lambda^0/\Lambda^S)^*$ and the index $[\Lambda^0 : \Lambda^S] = m^2$, we know that the subgroup S for $\Lambda_{\mathrm{I}}(m)$ has m^2 elements. As $S_{\mathrm{I}}(m)$ consists of m^2 elements of T that vanishes on $\Lambda^S_{\mathrm{I}}(m)$, the subgroup corresponding to $\Lambda^S_{\mathrm{I}}(m)$ must be $S_{\mathrm{I}}(m)$. □

Let $S_r = \{t \in T : t^r t^{-1} \in S\}$ and $S_s = \{s \in T : t^s t^{-1} \in S\}$. They are subgroups of T.

Proposition 3.6. *We have the formula $S^+ = S_r \cap S_s$.*

Proof. The containment $S^+ \subset S_r \cap S_s$ is obvious. Suppose now $t \in S_r \cap S_s$ and $v, w \in W$. Then $t^{vw}t^{-1} = (t^v t^{-1})^w t^w t^{-1}$ shows $S^+ \supset S_r \cap S_s$. □

The formula also works in more general situations where W is not necessarily D_8 with the obvious modifications. In principle, one can calculate the Weyl groups now and it is not a difficult calculation. We will now describe the second approach via lattices instead of working with the subgroups directly, which is slightly nicer.

Let $\Lambda^S_r = \Lambda^{S_r}$ and $\Lambda^S_s = \Lambda^{S_s}$.

Proposition 3.7. *We have the formula $\Lambda^S_+ = \langle \Lambda^S_r, \Lambda^S_s \rangle$, which is the abelian group generated by Λ^S_r and Λ^S_s.*

Proof. This follows from proposition 3.6 and the fact $\Lambda^- : \mathbf{mod}\text{-}W(T) \to W\text{-}\mathbf{mod}(\Lambda^0)$ is an anti-isomorphism. $\square$

Let $\Lambda^0 = \mathbb{Z}^2$. We pick a basis

$$\lambda_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

for Λ^S. Let $X = (\lambda_1 \quad \lambda_2)$ be the 2×2 matrix with λ_1, λ_2 as columns. We shall represent elements of $S = S^{**}$ as row vectors and write the same row vector for both s and s^{**}. For $t = (a \quad b) \in T = (\mathbb{R}/\mathbb{Z})^2$, we have $t \in S$ if and only if $tX = 0 \in (\mathbb{R}/\mathbb{Z})^2$. Alternatively, we could require that $tX \in \mathbb{Z}^2$.

We can now describe Λ^S_r and Λ^S_s using linear algebra. Consider the matrices

$$M_r = \rho(r) - I_2 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad M_s = \rho(s) - I_2 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

Proposition 3.8 ([4], p. 12)**.** *We have the equalities $\Lambda^S_r = M_r\Lambda^S$ and $\Lambda^S_s = M_s\Lambda^S$.*

We need a linear algebra lemma. For every $u, v \in \mathbb{R}^m$, we write $u \cdot v$ for their dot product. If $S \subset \mathbb{R}^m$, we define $u \cdot S = \{u \cdot v : v \in S\}$.

Lemma 3.9. *Let $A, B \subset \mathbb{Z}^m$. The condition $\langle A\rangle_{\mathbb{Z}} = \langle B\rangle_{\mathbb{Z}}$ is equivalent to the condition that for every $u \in \mathbb{R}^{1\times m}$, we have $u \cdot A \subset \mathbb{Z}$ if and only if $u \cdot B \subset \mathbb{Z}$. By $\langle A\rangle_{\mathbb{Z}}$, we mean the $\mathbb{Z}$-span of A.*

The proof is straightforward linear algebra. The details are given in the following note [6].

Proof of 3.8. We will only show that $\Lambda^S_r = M_r\Lambda^S$.

Pick any basis $A = \{\mu_1, \mu_2\}$ of Λ^S_r. By proposition 3.4, $tA \subset \mathbb{Z}$ if and only if $t \in S_r$. For any row vector $v \in \mathbb{Z}^2$ we have $tA \subset \mathbb{Z}$ if and only if $(t + v)A \subset \mathbb{Z}$.

By the definition of S_r, an element $t = (a \quad b) \in T$ is in S_r if and only if $t\rho(r) - t$ vanishes on Λ^S. Equivalently, $t \in S_r$ if and only if $tM_rX \in \mathbb{Z}^2$. Let $B = \{M_r\lambda_1, M_r\lambda_2\}$. Again, for any row vector $v \in \mathbb{Z}^2$, we have $tB \subset \mathbb{Z}$ if and only if $(t + v)B \subset \mathbb{Z}$.

By lemma 3.9, we have $\Lambda^S_r = \langle A\rangle_{\mathbb{Z}} = \langle B\rangle_{\mathbb{Z}} = M_r\Lambda^S$. $\square$

Proposition 3.10. *For each $\Lambda^S \neq 0$, we have $W_G(H) = \Lambda^S/\Lambda^S_+ \cong C_2$.*

Proof. Since $M_s\Lambda^S_{\mathrm{I}}(m) \subset M_r\Lambda^S_{\mathrm{I}}(m) = \Lambda^S_{\mathrm{II}}(m)$, the lattice $\Lambda^S_{\mathrm{I},+}(m)$ is actually $\Lambda^S_{\mathrm{II}}(m)$. Similarly, $M_s\Lambda^S_{\mathrm{II}}(m) \subset M_r\Lambda^S_{\mathrm{II}}(m) = \Lambda^S_{\mathrm{I}}(2m) = \Lambda^S_{\mathrm{II},+}(m)$. Note that $[\Lambda^S_{\mathrm{I}}(m) : \Lambda^S_{\mathrm{II}}(m)] = [\Lambda^S_{\mathrm{II}}(m) : \Lambda^S_{\mathrm{I}}(2m)] = 2$, so we get $W_G(H) \cong C_2$ for all subgroups $H(S, \tau)$. $\square$

4. Topology

In the first half of this section, we define the topology on $\mathfrak{X}_{\text{full}}$. Then we make explicit calculations in the second half to determine a neighborhood basis at each point of $\mathfrak{X}_{\text{full}}$.

We can endow $T = (\mathbb{R}/\mathbb{Z})^2$ with the T-bi-invariant metric $d_T(t_1, t_2) = \min \left\| \widetilde{t_1} - \widetilde{t_2} \right\|_\infty$ where $\widetilde{t_1}, \widetilde{t_2}$ varies over all possible lifts of t_1, t_2 to $\mathbb{R}^2$. We can extend it to a G-bi-invariant metric d on G as follows.

Write $G = \coprod_{w \in W} \sigma(w) T$. Define

$$d(\sigma(w)t_1, \sigma(v)t_2) = \begin{cases} d_T(t_1, t_2) & \text{if } v = w; \\ 5 & \text{if } v \neq w. \end{cases}.$$

We picked the number 5 only because $5 > d(t_1, t_2)$ for each pair $t_1, t_2 \in T$. One can then check that d satisfies the triangle inequality and that it is G-bi-invariant by using the fact T is normal in G.

Let us denote the collection of closed (hence compact) subsets of G by $K(G)$. The Hausdorff distance d_H associated to d is defined as

$$d_H(K, L) = \max \left(\sup_{x \in K} d(x, L), \sup_{y \in L} d(K, y) \right)$$

for each $K, L \in K(G)$. This makes $K(G)$ into a compact metric space. The collection of closed subgroups $\text{Sub}(G)$ of G is then a metric subspace of $K(G)$.

The underlying topological space of $K(G)$ has the Vietoris topology, which only depends on the topology of G, not on the metric we chose [8, p. 67]. We could have started with a different metric on G, not even necessarily a bi-invariant one and we would have still obtained the same topological space $\text{Sub}(G)$.

We define the topological space $\text{Sub}(T)$ for the maximal torus in a similar fashion. The collection of W-invariant subgroups W-$\text{Sub}(T)$, which is the set of objects of **mod-**$W(T)$, can be given the subspace topology from $\text{Sub}(T)$.

Let $\mathfrak{X} = \text{Sub}(G)/G$ be the collection of conjugacy classes of G. We give it the quotient topology under the canonical map $\text{Sub}(G) \to \text{Sub}(G)/G$. We denote by $\text{Sub}(G)_{\text{full}}$ the subspace of $\text{Sub}(G)$ consisting of full subgroups. Similarly, $\mathfrak{X}_{\text{full}}$ is the subspace of $\mathfrak{X}$ consisting of conjugacy classes coming from full subgroups.

There is a continuous map $\text{Sub}(G)_{\text{full}} \to W$-$\text{Sub}(T)$ sending a subgroup $H(S, \tau)$ to S. This map factors through the canonical map $\text{Sub}(G)_{\text{full}} \to \mathfrak{X}_{\text{full}}$. Here is a commutative diagram summarizing the situation.

$$\begin{array}{ccc} \mathrm{Sub}(G)_{\mathrm{full}} & \xrightarrow{p} & \mathfrak{X}_{\mathrm{full}} \\ & \searrow^{q} & \downarrow{\scriptstyle \bar{q}} \\ & & W\text{-}\mathrm{Sub}(T) \end{array}$$

Now we discuss some point-set topology properties of the spaces and maps we have in the diagram above. Most of these are special cases of basic results in transformation groups and the proofs are routine. For brevity, we shall omit most proofs and point to suitable references.

Proposition 4.1 ([3], p. 108)**.** *The conjugation action $G \times Sub(G)_{full} \to Sub(G)_{full}$ is continuous.*

Proposition 4.2 ([1], p. 38)**.** *The space $\mathfrak{X}_{full}$ is Hausdorff.*

Proposition 4.3. *The map p is open.*

Proof. The proposition follows from the equality $U = \bigcup_{g \in G} g^{-1}Ug$ for any open set $U \subset \mathrm{Sub}(G)_{\mathrm{full}}$. □

We shall now describe the topology of $\mathfrak{X}_{\mathrm{full}}$ by giving a neighborhood basis at each point. We need the fundamental result of Montgomery and Zippin.

Proposition 4.4 (Montgomery-Zippin)**.** *Consider a compact Lie group L, a closed subgroup H of L and a neighborhood U of the identity e. Then there is a neighborhood $W \subset U$ of e such that for each subgroup $K \subset WH$ there is a $u \in U$ so that $u^{-1}Ku \subset H$.*

The proof is given in [1, p. 87]. To clarify, the set WH is obtained by multiplying elements of W and H, so it should be thought of as a "W-thickening" of H.

Lemma 4.5 ([4], lemma 5.3)**.** *If $S \subset T$ is a subgroup, we can find a neighborhood of S consisting only of subgroups $S' \subset S$.*

Proof. Since T is itself a neighborhood of the identity in G, the lemma holds by Montgometry-Zippin and the fact that T is abelian. □

Lemma 4.6. *For a full subgroup $H(S, \tau)$ of G, $d_H(H, G) = d_H(S, T)$.*

Proof. Note that $H(S, \tau) \subset G$ implies $d_H(H, G) = \sup_{g \in G} d(H, g)$. Fix a point $g \in G$. There is a point $h \in H$ lying in the same component as g such that $d(h, g) = d(H, g)$ since the distance between components is much larger than the distance between points in the same component.

By the bi-invariance of d, we can assume that $g \in T$ and we have $d(H, g) = d(S, g)$. This proves the lemma because $d(S, T) = \sup_{g \in T} d(S, g)$. □

Now we give a quantitative result on distance.

Lemma 4.7. $d(S_{\mathrm{I}}(m), T) = d(S_{\mathrm{II}}(m), T) = \frac{1}{2m}$.

Proof. The kernel of $\mathbb{R}^2 \to T/S_{\mathrm{I}}(m)$ is $\mathbb{Z}[\frac{1}{m}]^2$. By the definition of $\|\cdot\|_\infty$, any lift of $t \in T$ is within $\frac{1}{2m}$ of $\mathbb{Z}[\frac{1}{m}]^2$ and this distance can be achieved.

Similarly, the kernel of $\mathbb{R}^2 \to T/S_{\mathrm{II}}(m)$ is $\mathbb{Z}[\frac{1}{m}]^2 \cup (\frac{1}{2m}, \frac{1}{2m}) + \mathbb{Z}[\frac{1}{m}]^2$. The equality $d(S_{\mathrm{II}}(m), T) = \frac{1}{2m}$ follows. □

We define for each integer $n \geq 1$ a set U_n consisting of all conjugacy classes $[H(S, \tau)]$ where $S = S_{\mathrm{I}}(m)$ or $S = S_{\mathrm{II}}(m)$ with $m \geq n$.

Proposition 4.8. *(1) For each $[H(S,\tau)] \in \mathfrak{X}_{full}$ where $S \neq T$, the collection $\mathcal{N}([H(S,\tau)]) = \{\{[H(S,\tau)]\}\}$ is a neighborhood basis.*
(2) For the point $[G]$, the collection $\mathcal{N}([G]) = \{U_n : n \geq 0\}$ is a neighborhood basis.

Proof. Take a subgroup $S \neq T$ in W-Sub(T). By lemma 4.5, it has a neighborhood consisting of $S' \subset S$. If $S' \neq S$, $d(S', S) > 0$ because S' is finite. There are only finitely many subgroups S' of S, so the singleton $\{S\}$ is open in W-Sub(T).

Suppose point $[H_1] \in \mathfrak{X}_{\mathrm{full}}$ maps to $S \in W$-Sub(T). Then $\overline{q}^{-1}(\{S\})$ consists of two points in $\mathfrak{X}_{\mathrm{full}}$, one of them being $[H_1]$. Let us call the other $[H_2]$. Since $\mathfrak{X}_{\mathrm{full}}$ is Hausdorff, we can find disjoint open set $[H_1] \in U$ and $[H_2] \in V$. Now $\{[H]\} = U \cap \{[H_1], [H_2]\}$ is open, so we have shown (1).

The inverse image of the point $[G] \in \mathfrak{X}_{\mathrm{full}}$ consists of the point $G \in \mathrm{Sub}(G)_{\mathrm{full}}$ alone. Take a sequence of real numbers α_m such that $\frac{1}{2} < \alpha_1 < 1$ and $\frac{1}{2(m+1)} < \alpha_{m+1} < \frac{1}{2m}$. The open balls $B_{\alpha_n}(G) = \{H(S,\tau) : d(S,T) < \alpha_n\}$ form a neighborhood basis at G. Since $U_n = p\left(B_{\alpha_n}(G)\right)$ and p is an open map, the collection $\mathcal{N}([G])$ is a neighborhood basis for $[G]$.

□

References

[1] G.E. Bredon. **Introduction to Compact Transformation Groups**. Pure and Applied Mathematics. Academic Press, 1972. ISBN 9780080873596. URL https://books.google.co.uk/books?id=qXEl_AIeSBUC.

[2] K.S. Brown. **Cohomology of Groups**. Graduate Texts in Mathematics. Springer New York, 2012. ISBN 9781468493276. URL https://books.google.co.uk/books?id=2fzlBwAAQBAJ.

[3] T.t. Dieck. **Transformation Groups and Representation Theory**. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2006. ISBN 9783540385172. URL https://books.google.co.uk/books?id=IOd7CwAAQBAJ.

[4] J.P.C. Greenlees. Spaces of subgroups of toral groups. **arXiv preprint arXiv:2501.06914**, 2025.

[5] C. Gerig (https://mathoverflow.net/users/12310/chris gerig). Kuenneth-formula for group cohomology with nontrivial action on the coefficient. MathOverflow. URL https://mathoverflow.net/q/75485. URL:https://mathoverflow.net/q/75485 (version: 2013-07-11).

[6] Z. Huang. A criterion for determining the same lattice. Github. URL https://zihengh63.github.io/Expository_notes/Modules/A_criterion_for_detecting_the_same_lattice.pdf.

[7] S.A. Morris. **Pontryagin Duality and the Structure of Locally Compact Abelian Groups**. London Mathematical Society Lecture Note Series. Cambridge University Press, 1977.

[8] S.M. Srivastava. **A Course on Borel Sets**. Graduate Texts in Mathematics. Springer New York, 1998. ISBN 9780387984124. URL https://books.google.co.uk/books?id=FhYGYJtMwcUC.