Fundamental group of the circle via groupoids

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Acknowledgment

The present essay was written as part of the module MA262 Scientific Communication at Warwick. I would like to thank José Rodrigo for supervising the essay and for his helpful advice.

Contents

Introduction

Consider the problem of distinguishing topological spaces. Take the unit circle $S^1 \subset \mathbb{C}$ and the unit sphere $S^2 \subset \mathbb{R}^3$ as examples. They certainly look different. However, how do we know that S^1 and S^2 are not homeomorphic?

Algebraic topology approaches this problem by systematically assigning "algebraic invariants" to topological spaces. These invariants are objects that encode certain kind of information about the space. Looking at invariants brings into focus the relevant properties that make topological spaces different. In section 1, we introduce the formal framework for discussing algebraic invariants.

For instance, given a topological space X, a basic invariant we can look at is the fundamental group $\pi_1(X, x_0)$ based at a point $x_0 \in X$. The group $\pi_1(X, x_0)$ captures information about loops starting and ending at x_0. Our goal is to work out $\pi_1(S^1, 1)$, which is arguably the simplest nontrivial example.

Classically, $\pi_1(S^1, 1)$ is calculated using covering spaces [7, pp. 29-31]. In 1960s, Brown [1] made a curious discovery that led to an algebraic approach via the van Kampen theorem. This requires using a slightly different invariant: fundamental groupoids. The construction of these algebraic invariants and the van Kampen theorem are the subjects of section 2.1 and section 2.2 respectively. Finally, we complete our promise and calculate $\pi_1(S^1, 1)$ in section 2.3.

1 Categorical Preliminaries

The language of categories and functors gives us a way to formalize the idea of algebraic invariants. We can discuss formal properties and manipulations of algebraic invariants thanks to this language. In section 2.1, we shall be building algebraic models of topological spaces as categories.

Most of the definitions and results are standard [10].

1.1 Categories

Let us start off with the definition of a category and then we will see some examples of categories.

A **category** $\mathcal{C}$ consists of a collection of objects and for every pair of objects a collection of morphisms (or maps) satisfying some composition rules. The collection of objects in $\mathcal{C}$ is denoted $\mathrm{Ob}\,\mathcal{C}$. If $x, y \in \mathrm{Ob}\,\mathcal{C}$, the collection of morphisms from x to y is denoted $\mathcal{C}(x, y)$. An alternative notation for $\mathcal{C}(x, y)$ is $\mathrm{Hom}_{\mathcal{C}}(x, y)$ which is why we refer to it as the **hom-set**. For every $f \in \mathcal{C}(x, y)$, the **domain** and **codomain** of f are x and y respectively. They are also part of the data of a morphism, which means two morphisms cannot be equal if their domains or codomains do not match. We require the following:

- for every $x, y, z \in \mathrm{Ob}\,\mathcal{C}$, there is a map $\circ : \mathcal{C}(x, y) \times \mathcal{C}(y, z) \to \mathcal{C}(x, z)$ sending the pair $(f, g) \in \mathcal{C}(x, y) \times \mathcal{C}(y, z)$ to a morphism $g \circ f \in \mathcal{C}(x, z)$. The map $\circ$ is called composition. We write gf for $g \circ f$ when no confusion will arise.

- given any three morphisms f, g, h such that hg and gf are both defined, we have $(hg)f = h(gf)$. In other words, composition is associative.
- for each $x \in \operatorname{Ob}\mathcal{C}$, we have an **identity morphism** $\mathrm{id}_x \in \mathcal{C}(x,x)$ satisfying

$$f \circ \mathrm{id}_x = f \text{ and } \mathrm{id}_x \circ g = g$$

  whenever $y \in \operatorname{Ob}(\mathcal{C})$, $f \in \mathcal{C}(x,y)$ and $g \in \mathcal{C}(y,x)$.

Some comments are in order. We use "collection" instead of "sets" because we do not require $\operatorname{Ob}\mathcal{C}$ or $\mathcal{C}(x,y)$ to be sets. It can be confusing because a "hom-set" may not be a set in our convention. If all hom-sets in a category are sets, we say the category is locally small. In [8], local smallness is implicitly assumed in the definition of a category. The foundational issues that come with sizes are a topic in itself and there are different approaches to setting a foundation for categories [11]. Fortunately these issues do not crop up in our discussions, so we shall make no more mention of them. We will use set-theoretic notations such as "$x \in \operatorname{Ob}\mathcal{C}$" and "$\operatorname{Ob}\mathcal{C} \cup \operatorname{Ob}\mathcal{D}$" with no qualms.

We list some examples of categories in the following table.

Name of Category	Objects	Morphisms
Top	topological spaces	continuous functions
Set	sets	set functions
Grp	groups	group homomorphisms

Let $\operatorname{Mor}\mathcal{C}$ be the collection of all morphisms in a category $\mathcal{C}$, i.e. $\{f \in \mathcal{C}(x,y) : x, y \in \mathcal{C}\}$. We say a category $\mathcal{C}$ is **small** if $\operatorname{Mor}\mathcal{C}$ forms a set. This implies $\operatorname{Ob}\mathcal{C}$ forms a set since we can identify objects with their identity morphisms. In fact, categories can be defined in an object-free fashion [6, p. 5].

There are a few kinds of categories that we will use as indexing categories. An indexing category is like an indexing set, except it has morphisms, so it is a step towards formalizing the idea of diagrams, which are pervasive in category theory. For our purposes, an indexing category is simply a small category.

Example 1.1. A **discrete category** is a category with only identity morphisms, so it contains no more data than its underlying collection of objects.

Example 1.2. A poset $(P, \leq)$ is a small category with the elements of the poset as objects and a unique morphism $x \to y$ between $x, y \in P$ if and only if $x \leq y$.

1.2 Functors

Given two categories $\mathcal{C}$ and $\mathcal{D}$, a **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of the following:

- a map $\operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$ that sends each $c \in \operatorname{Ob}\mathcal{C}$ to some $Fc \in \operatorname{Ob}\mathcal{D}$,
- for each pair $c, c' \in \operatorname{Ob}\mathcal{C}$ a map $\mathcal{C}(c,c') \to \mathcal{D}(Fc, Fc')$ sending each $f \in \mathcal{C}(c,c')$ to some $F(f) = Ff \in \mathcal{D}(Fc, Fc')$,

preserving the structure of $\mathcal{C}$ in the following sense:

- $\forall c \in \operatorname{Ob}\mathcal{C}$ we have $F(\mathrm{id}_c) = \mathrm{id}_{Fc}$,
- if $f : c \to c'$ and $g : c' \to c''$ are morphisms in $\mathcal{C}$, we have $F(g \circ f) = Fg \circ Ff$.

Two functors are equal if they agree on objects and hom-sets of every pair of objects. Functors are everywhere so we will see examples in due course.

An incredibly useful feature of functors is that they preserve isomorphisms.

Definition 1.1. A morphism $f : c \to c'$ in $\mathcal{C}$ is an **isomorphism** if there is $g : c' \to c$ such that $gf = \mathrm{id}_c$ and $fg = \mathrm{id}_{c'}$. We write $f : c \cong c'$ in this case.

Proposition 1.1. *Let $f : c \to c'$ be an isomorphism in $\mathcal{C}$. Then $Ff : Fc \to Fc'$ is an isomorphism in $\mathcal{D}$.*

The proposition follows from the definition of a functor [10, p. 18]. With functors and isomorphisms introduced, we can now explain what a groupoid is.

Definition 1.2. A **groupoid** G is a small category in which every morphism is an isomorphism. Groupoids with functors between them as morphisms form a category denoted **Grpd**.

Example 1.3. A small discrete category is a groupoid.

Example 1.4. There is a functor $B : \mathbf{Grp} \to \mathbf{Grpd}$ that sends each group G to the groupoid BG with a single object $*$. The morphisms in the groupoid BG are the elements of G with composition given by multiplication in the group. The identity of the group is the identity morphism. Since every element of the group is invertible, the category BG is indeed a groupoid. A group homomorphism $G \to H$ is exactly the same as a functor $BG \to BH$. Therefore, any group G can be regarded as a groupoid (and therefore a category!) via the functor B.

1.3 Natural Transformations

Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A **natural transformation** $\eta : F \Rightarrow G$ consists of a morphism $\eta_c : Fc \to Gc$ for each object c in $\mathcal{C}$ such that whenever $f : c \to c'$ is a morphism in $\mathcal{C}$, we have the commutative diagram below.

$$\begin{array}{ccc} Fc & \xrightarrow{\eta_c} & Gc \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fc' & \xrightarrow{\eta_{c'}} & Gc' \end{array}$$

A alternative notation we will use for $\eta : F \Rightarrow G$ is $\eta : F \simeq G$. The morphism η_c at the object c is said to be the component of η at c.

Example 1.5. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The objects of the **functor category** $\mathcal{D}^{\mathcal{C}}$ are functors $\mathcal{C} \to \mathcal{D}$ and the morphisms are natural transformations.

Definition 1.3. Two functors $F, G : \mathcal{C} \to \mathcal{D}$ are **naturally isomorphic** if they are isomorphic as objects in the functor category $\mathcal{D}^{\mathcal{C}}$. The natural transformation η that gives the isomorphism between the two functors is called a **natural isomorphism** and we write $\eta : F \cong G$ or simply $F \cong G$.

An equivalent way to express $\eta : F \cong G$ is that each component of the natural transformation η is an isomorphism.

Definition 1.4. A functor $F : \mathcal{C} \to \mathcal{D}$ is an **equivalence of categories** if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that $GF \cong \mathrm{id}_{\mathcal{C}}$ and $FG \cong \mathrm{id}_{\mathcal{D}}$. The categories $\mathcal{C}$ and $\mathcal{D}$ are said to be equivalent if such F and G exist between them.

We give another characterization of an equivalence of categories.

Definition 1.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. It is said to be

- **full** if the map $\mathcal{C}(c, c') \to \mathcal{D}(Fc, Fc')$ is surjective $\forall c, c' \in \mathrm{Ob}\,\mathcal{C}$;
- **faithful** if the map $\mathcal{C}(c, c') \to \mathcal{D}(Fc, Fc')$ is injective for $\forall c, c' \in \mathrm{Ob}\,\mathcal{C}$;
- **fully faithful** if it is both full and faithful;
- **essentially surjective** if $\forall d \in \mathrm{Ob}\,\mathcal{D}$, $\exists c \in \mathrm{Ob}\,\mathcal{C}$ such that $Fc \cong d$.

Proposition 1.2. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Proof. Suppose that we have a functor $F : \mathcal{C} \to \mathcal{D}$ and a functor $G : \mathcal{D} \to \mathcal{C}$ such that there are natural isomorphisms $\eta : \mathrm{id}_{\mathcal{C}} \cong GF$ and $\mu : \mathrm{id}_{\mathcal{D}} \cong FG$. We show that F is fully faithful and essentially surjective.

Let $c, c' \in \mathrm{Ob}\,\mathcal{C}$. We would like to show that $\mathcal{C}(c, c') \to \mathcal{D}(Fc, Fc')$ is injective. Suppose $f, g : c \to c'$ are morphisms such that $Ff = Fg$. Then $GFf = GFg$ and we have the following commutative diagrams

$$\begin{array}{ccc} c & \xrightarrow{\eta_c} & GFc \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GFf} \\ c' & \xrightarrow{\eta_{c'}} & GFc' \end{array} \qquad\qquad \begin{array}{ccc} c & \xrightarrow{\eta_c} & GFc \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle GFf = GFg} \\ c' & \xrightarrow{\eta_{c'}} & GFc' \end{array}$$

Thus, $f = g$ and $\mathcal{C}(c, c') \to \mathcal{D}(Fc, Fc')$ is injective. Similarly, $\mathcal{D}(d, d') \to \mathcal{D}(Gd, Gd')$ is injective for any $d, d' \in \mathrm{Ob}\,\mathcal{D}$.

To see the surjectivity of $\mathcal{C}(c, c') \to \mathcal{D}(Fc, Fc')$, let $g : Fc \to Fc'$ be a morphism. We define a morphism $f = \eta_{c'}^{-1} \circ Gg \circ \eta_c$ in $\mathcal{C}$ so that the diagram

$$\begin{array}{ccc} c & \xrightarrow{\eta_c} & GFc \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Gg} \\ c' & \xrightarrow{\eta_{c'}} & GFc' \end{array}$$

commutes by definition. Since $\eta : \mathrm{id}_{\mathcal{C}} \cong GF$ is a natural isomorphism, we also have the commutativity of the diagram

$$\begin{array}{ccc} c & \xrightarrow{\eta_c} & GFc \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GFf} \\ c' & \xrightarrow{\eta_{c'}} & GFc' \end{array}$$

The map $\mathcal{D}(Fc, Fc') \to \mathcal{C}(GFc, GFc')$ is injective, so $Ff = g$. This shows that $\mathcal{C}(c, c') \to \mathcal{D}(Fc, Fc')$ is surjective. The functor F is essentially surjective because $d \cong FGd$ via μ_d for any $d \in \operatorname{Ob}\mathcal{D}$. This completes one implication.

Now, suppose that $F : \mathcal{C} \to \mathcal{D}$ is a fully faithful, essentially surjective functor. For this direction, we need to invoke the axiom of choice. By essential surjectivity, we can choose for each $d \in \operatorname{Ob}\mathcal{D}$ an object $c \in \operatorname{Ob}\mathcal{C}$ and an isomorphism $\mu_d : d \cong Fc$. We define the functor G on objects by $G(d) = c$.

Suppose that we have another object d' in $\mathcal{D}$ with the isomorphism $\mu_{d'}$: $d' \cong Fc'$. Let $g : d \to d'$ be a morphism in $\mathcal{D}$. There is a morphism, namely $\mu_{d'} \circ g \circ \mu_d^{-1}$, that makes the diagram below commute.

$$\begin{array}{ccc} d & \xrightarrow{\mu_d} & Fc \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mu_{d'}\circ g\circ\mu_d^{-1}} \\ d' & \xrightarrow{\mu_{d'}} & Fc' \end{array}$$

Since F is a fully faithful functor, there is a unique morphism $f : c \to c'$ such that $Ff = \mu_{d'} \circ g \circ \mu_d^{-1}$. This means that we can define G on morphisms by $Gg = f$. We now need to prove that G is in fact a functor. Consider the morphism id_d in $\mathcal{D}$. Then $\mu_d \circ \mathrm{id}_d \circ \mu_d^{-1} = \mathrm{id}_{Fc} = F\mathrm{id}_c$, so $G\mathrm{id}_d = \mathrm{id}_c = \mathrm{id}_{Gd}$.

Suppose that we have the commutative diagram

$$\begin{array}{ccc} d & \xrightarrow{\mu_d} & Fc \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mu_{d'}\circ g\circ\mu_d^{-1}} \\ d' & \xrightarrow{\mu_{d'}} & Fc' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle \mu_{d''}\circ g'\circ\mu_{d'}^{-1}} \\ d'' & \xrightarrow{\mu_{d''}} & Fc'' \end{array}$$

in $\mathcal{D}$. The outer rectangle forms the commutative square

$$\begin{array}{ccc} d & \xrightarrow{\mu_d} & Fc \\ {\scriptstyle g'\circ g}\downarrow & & \downarrow{\scriptstyle \mu_{d''}\circ(g'\circ g)\circ\mu_d^{-1}} \\ d'' & \xrightarrow{\mu_{d''}} & Fc'' \end{array}$$

which upon comparison gives $G(g' \circ g) = Gg' \circ Gg$.

The isomorphisms $\mu_d : d \to Fc = FGd$ form a natural isomorphism μ : $\mathrm{id}_{\mathcal{D}} \cong FG$ since each $g : d \to d'$ induces $FGg = \mu_{d'} \circ g \circ \mu_d^{-1}$ by definition.

The last thing we need to prove is that we in fact have a natural isomorphism $\eta : \mathrm{id}_{\mathcal{C}} \cong GF$. Suppose that c_0 is an object of $\mathcal{C}$. We need an isomorphism $\eta_{c_0} : c_0 \cong GFc_0$. Consider the isomorphism $\mu_{Fc_0} : Fc_0 \cong Fc$. We have a unique morphism $\eta_{c_0} : c_0 \to c = GFc_0$ since F is fully faithful.

The morphism η_{c_0} is an isomorphism also thanks to F being fully faithful. Indeed, there is a unique morphism $\varepsilon : Fc \to Fc_0$ such that $F\varepsilon = \mu_{Fc_0}^{-1}$. Then $F(\varepsilon \circ \eta_{c_0}) = \mathrm{id}_{Fc_0}$ and $F(\eta_{c_0} \circ \varepsilon) = \mathrm{id}_{Fc}$ show that $\varepsilon \circ \eta_{c_0} = \mathrm{id}_{c_0}$ and $\eta_{c_0} \circ \varepsilon = \mathrm{id}_c$.

We need to show that the collection of isomorphisms forms a natural isomorphism $\eta : \mathrm{id}_{\mathcal{C}} \cong GF$. Let $f : c_0 \to c_0'$ be a morphism in $\mathcal{C}$. There is a commutative diagram

$$\begin{array}{ccc} Fc_0 & \xrightarrow{\mu_{Fc}} & Fc \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle \mu_{Fc_0'}\circ Ff\circ\mu_{Fc}^{-1}} \\ Fc_0' & \xrightarrow{\mu_{Fc_0'}} & Fc' \end{array}$$

We defined GFf to be the morphism such that $FGFf = \mu_{Fc_0'} \circ Ff \circ \mu_{Fc}^{-1}$. Since F is fully faithful, we have a commutative square

$$\begin{array}{ccc} c_0 & \xrightarrow{\eta_{c_0}} & GFc \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GFf} \\ c_0' & \xrightarrow{\eta_{c_0'}} & GFc' \end{array}$$

which shows that η is natural. This completes the proof. □

Definition 1.6. A **subcategory** $\mathcal{D}$ of $\mathcal{C}$ is a category consisting of:

- a subcollection $\mathrm{Ob}\,\mathcal{D} \subset \mathrm{Ob}\,\mathcal{C}$ of objects,
- a subcollection of morphism $\mathcal{D}(d, d') \subset \mathcal{C}(d, d')$ for each pair $d, d' \in \mathrm{Ob}\,\mathcal{D}$,
- a composition rule that is induced by the composition rule of $\mathcal{C}$.

Another way one could define a category $\mathcal{D}$ to be a subcategory of $\mathcal{C}$ is to say that there is a faithful functor $\mathcal{D} \to \mathcal{C}$ that is injective on objects [10, p. 31]. This is not standard, but it captures all the data above rather succinctly.

Definition 1.7. Suppose $\eta : F \Rightarrow G$ is a natural transformation between two functors $F, G : \mathcal{C} \to \mathcal{C}'$ and $\mathcal{D}$ is a subcategory of $\mathcal{C}$. We write $F \simeq G \,\mathrm{rel}\, \mathcal{D}$ if all components of η at objects in $\mathcal{D}$ are identity morphisms.

Definition 1.8. Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$ and $i : \mathcal{D} \to \mathcal{C}$ the standard inclusion functor. We say that $\mathcal{D}$ is a **deformation retract** of $\mathcal{C}$ if there is a functor $r : \mathcal{C} \to \mathcal{D}$ with $ir \cong \mathrm{id}_{\mathcal{C}} \,\mathrm{rel}\, \mathcal{D}$. The functor r is a deformation retraction.

A deformation retraction is a retraction in the sense that $ri = \mathrm{id}_{\mathcal{D}}$. The condition $ir \cong \mathrm{id}_{\mathcal{C}} \,\mathrm{rel}\, \mathcal{D}$ implies $iri = i$ as functors. For any $d \in \mathrm{Ob}\,\mathcal{D}$, $iri(d) = i(d)$ implies $ri(d) = d$ as i is injective on objects. Similarly, i is faithful shows $ri(f) = f$ for any morphism f in $\mathcal{D}$. The equality $ri = \mathrm{id}_{\mathcal{D}}$ implies $ri \cong \mathrm{id}_{\mathcal{D}}$ if we take the natural isomorphism to be the identity natural transformation.

Corollary. *Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$. The standard inclusion $\mathcal{D} \to \mathcal{C}$ is a deformation retraction if and only if it is fully faithful and essentially surjective.*

Remark 1. We omit the proof of the corollary. It is not a direct consequence of proposition 1.2, but rather the proof of it. We can get a deformation retraction instead of an equivalence of categories in this case because the inclusion of a subcategory is injective on objects. We may therefore choose the components of the natural transformation at objects in $\mathcal{D}$ to be identity morphisms.

1.4 Colimits

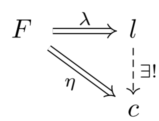
Let $\mathcal{J}$ be a small category which will serve as an indexing category. As preparation for the definition of a colimit, we formalize the idea of diagrams first.

Definition 1.9. A **diagram** in $\mathcal{C}$ is a functor from an indexing category to $\mathcal{C}$.

For any object c in a category $\mathcal{C}$, we have a constant functor from $\mathcal{J} \to \mathcal{C}$ that sends every object in $\mathcal{J}$ to c and every morphism in $\mathcal{J}$ to id_c. A constant functor at an object $c \in \mathrm{Ob}\,\mathcal{C}$ will be also denoted by c.

Definition 1.10. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram. A **cocone** under F is a natural transformation $\eta : F \Rightarrow c$ where $c : \mathcal{J} \to \mathcal{C}$ is a constant functor.

Definition 1.11. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram. A **colimit** of F is a universal cocone $\lambda : F \Rightarrow l$ under F if it exists. This means that whenever $\eta : F \Rightarrow c$, we have the following commutative diagram.



A natural transformation between two constant functors is just a morphism of the objects defining the respective functors. If two objects both give rise to a colimit for the same diagram, these two objects must be isomorphic. This justifies the usage of *the* colimit. In the diagram above, we write $l = \mathrm{colim}_{\mathcal{J}} F$.

Colimits provide a convenient way to express the idea of "gluing" because an indexing category usually has some nonidentity morphisms which pose extra conditions on the colimit.

Example 1.6. Let $\mathcal{J}$ be the category that has three objects and two nonidentity morphisms $\bullet \leftarrow \bullet \rightarrow \bullet$. A **pushout** in $\mathcal{C}$ is the colimit of a diagram $\mathcal{J} \to \mathcal{C}$.

2 The Van Kampen theorem

2.1 Homotopy and the fundamental groupoid

It is standard to denote the unit interval $[0, 1] \subset \mathbb{R}$ by I.

Definition 2.1. Two continuous functions $f, g : X \to Y$ are **homotopic** if there is a continuous function $H : X \times I \to Y$ such that $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$. We write $H : f \simeq g$ and say that H is a homotopy from f to g.

Now we define the important notion of concatenation of homotopies.

Definition 2.2. Given two homotopies $H, K : X \times I \to Y$ such that $H|_{X\times\{1\}} = K|_{X\times\{0\}}$, their **concatenation** is the homotopy $K \cdot H$ given by the formula

$$(K \cdot H)(s,t) = \begin{cases} H(s, 2t) & \text{if } (s,t) \in X \times \left[0, \frac{1}{2}\right], \\ K(s, 2t-1) & \text{if } (s,t) \in X \times \left[\frac{1}{2}, 1\right]. \end{cases}$$

The concatenation of two homotopies does the first one at twice the speed and then the second one at twice the speed. We write the first homotopy on the right like function composition, but it is often written from left to right as well.

Definition 2.3. Let $f : X \to Y$ be continuous. We say that the homotopy $H : X \times I \to Y$ given by $H(s,t) = f(s)$ is a **constant homotopy**.

Definition 2.4. Let $H : X \times I \to Y$ be a homotopy. The **reverse** of H is the homotopy $R : X \times I \to Y$ given by $R(s,t) = H(s, 1-t)$ for all $(s,t) \in X \times I$.

The terminology "reverse" is not standard. We have the following immediate result [3, p. 225].

Proposition 2.1. *For any two topological spaces X, Y, the relation of homotopy is an equivalence relation on the set of continuous functions from X to Y.*

We can define a path in a topological space X as a homotopy p from a continuous function $x : I^0 \to X$ to a continuous function $y : I^0 \to Y$, where $I^0 = \{0\} \subset \mathbb{R}$ is the one-point space. We write $p : x \to y$ and say that the path starts at x and ends at y. Concatenation of two paths, constant paths and reverse paths are then special cases of the corresponding concept for homotopies.

We identify the function $x : I^0 \to X$ with the single point x in its image. The more general notions we defined for homotopies will still be needed in situations where the homotopy is not a path, so it is not for the sake of abstraction.

The concatenation of paths is not associative. However, it is associative up to homotopy. Here "homotopy" means homotopy relative to a subspace.

Definition 2.5. Let $A \subset X$ be a subspace of a topological space. A homotopy $H : X \times I \to Y$ from f to g whose restriction $H|_{A \times I}$ is a constant homotopy is said to be a **homotopy relative to** A. We write $H : f \simeq g \operatorname{rel} A$.

Proposition 2.2. *Let $p : x \to y$, $q : y \to z$, $r : z \to w$ be paths in X. Then $r(qp) \simeq (rq)p \operatorname{rel} \{x, w\}$.*

We can now construct the fundamental groupoid functor $\Pi_1 : \mathbf{Top} \to \mathbf{Grpd}$.

Given a topological space X, the objects in $\Pi_1(X)$ are the points in X. For two points x, y of X, the set of morphisms is the set of paths from x to y modulo the homotopy rel end points relation. That is, $p, q : x \to y$ represent the same morphism between x and y if $p \simeq q \operatorname{rel} \{x, y\}$.

Composition of morphisms in $\Pi_1(X)$ is given by concatenation of representative paths. Definition 2.2 shows it is well-defined. Constant paths represent the identity morphisms. We have associativity of morphisms by proposition 2.2, so $\Pi_1 X$ is a category. The reverse path construction shows that $\Pi_1 X$ is a groupoid.

A continuous function $f : X \to Y$ induces a functor $\Pi_1 f : \Pi_1 X \to \Pi_1 Y$ between groupoids sending each $x \in X$ to $f(x) \in Y$ and each path $p : x \to y$ to the path $f \circ p : f(x) \to f(y)$. Hence, $\Pi_1 : \mathbf{Top} \to \mathbf{Grpd}$ is a functor.

The fundamental groupoid we just defined is the "absolute" version. There is a relative version which will be the version we use later.

We have a category $\mathbf{Top}_{pair}$ with pairs of topological spaces (X, A), where $A \subset X$, as objects. A morphism from (X, A) to (Y, B) is a continuous function $f : X \to Y$ such that $f(A) \subset B$.

The fundamental groupoid of a pair (X, A) has as objects the set of points in A. The morphisms are paths in X (modulo the homotopy rel end points relation) which have end points in A. We denote it by $\Pi_1(X, A)$. This is a groupoid and the assignment $\Pi_1 : \mathbf{Top}_{pair} \to \mathbf{Grpd}$ is again a functor.

The category $\mathbf{Top}_{pair}$ has a subcategory $\mathbf{Top}_*$ where objects are (X, A) with A being a single point of X. The morphisms in $\mathbf{Top}_*$ are exactly the same as the morphisms in $\mathbf{Top}_{pair}$. The category $\mathbf{Top}_*$ is called the category of pointed topological spaces. The restriction of Π_1 to $\mathbf{Top}_*$ is denoted π_1. The functor π_1 is called the **fundamental group** because the fundamental groupoid based at a single point is indeed a group.

One may worry that the functor π_1 we defined lands in **Grpd** instead of **Grp**. Books and articles on the subject do not address this point probably because the concern seems pedantic. Here is a way to resolve the issue, but we do not adopt it because of cumbersome notation. We could define Π_1 as a functor from $\mathbf{Top}_{pair,*}$ to $\mathbf{Grpd}_*$. The objects in $\mathbf{Top}_{pair,*}$ are (X, A, x_0) with $x_0 \in A \subset X$ and the morphisms are continuous functions $f : (X, A, x_0) \to (Y, B, y_0)$ satisfying $f(A) \subset f(B)$ and $f(x_0) = f(y_0)$. An object of $\mathbf{Grpd}_*$ is a pointed groupoid: a pair $(\mathcal{G}, x)$ where x is an object (base point) of the groupoid $\mathcal{G}$. A morphism of $\mathbf{Grpd}_*$ is a functor between groupoids that preserves base points. There is a functor $\mathrm{Aut} : \mathbf{Grpd}_* \to \mathbf{Grp}$ sending $(\mathcal{G}, x)$ to $\mathrm{Aut}(x) = \mathrm{Hom}_{\mathcal{G}}(x, x)$ which is a group because $\mathcal{G}$ is a groupoid. A functor between pointed groupoids induces a group homomorphism between the automorphism groups at the base points. We may define π_1 to be Π_1 followed by Aut.

We now exhibit some more examples of groupoids and show some basic calculations of fundamental groupoids.

Definition 2.6. Let $E \subset \mathbb{R}^n$ be a convex set and $x, y \in E$. Let $p : x \to y$ be the straight line segment starting at x and ending at y. If $q : x \to y$ is a path in E, then we have the **straight line homotopy** $H : p \simeq q \,\mathrm{rel}\, \{x, y\}$ given by $H(s, t) = (1 - t)p(s) + tq(s)$ for all $(s, t) \in I \times I$.

Example 2.1. Consider the open interval $U = (-2, 2) \subset \mathbb{R}$ with the base points $U_0 = \{-1, 1\}$. Using the straight line homotopy, we see that there is exactly one morphism $s : -1 \to 1$ in $\Pi_1(U, U_0)$. Similarly, there is exactly one morphism $s^{-1} : 1 \to -1$. There are all the nontrivial morphisms of $\Pi_1(U, U_0)$.

We can generalize this example slightly. Since Π_1 is a functor, any pair of space homeomorphic to (U, U_0) produces the same (technically isomorphic) fundamental groupoid. We shall denote $\Pi_1(U, U_0)$ by $\mathcal{I}$ and depict the objects and nontrivial morphisms more abstractly as

$$* \underset{\longleftarrow}{\overset{\longrightarrow}{}} \bullet$$

For example, let $S^1 \subset \mathbb{C}$ be the circle. Then $\Pi_1(S^1 \setminus \{i\}, \{-1, 1\}) = \mathcal{I}$.

Example 2.2. Consider the union of two disjoint open intervals $U = (-2, 0) \cup (0, 2) \subset \mathbb{R}$ with base points $U_0 = \{-1, 1\} \subset U$. A morphism in $\Pi_1(U, U_0)$

starting at 1 can only end at 1 because -1 is not in the same path component as 1. If $p : 1 \to 1$ is a path, then the straight line homotopy shows that $p = \mathrm{id}_1$. Similarly for -1. This shows the fundamental groupoid $\Pi_1(U, U_0) = \Pi_1(S^1 \setminus \{i, -i\}, \{-1, 1\})$ is the discrete category consisting of two elements.

2.2 The van Kampen theorem

There are many formulations of the van Kampen theorem. We take the viewpoint that the van Kampen theorem allows us to compute the fundamental groupoid on a set of base points as the colimit on an open cover. In other words, we can break a complicated space up by choosing a "nice" open cover, compute the fundamental groupoid on each piece and glue them together to find the fundamental groupoid of the original space.

We need a definition in the set-up of the van Kampen theorem.

Definition 2.7. Let X be a topological space. A subset X_0 is **representative** in X if X_0 meets each path component of X.

Suppose $\mathcal{U} = \{U_\lambda\}$ is a finite open cover of a topological space X with the property that every finite intersection is again in $\mathcal{U}$. Let $X_0 \subset X$ be such that each $U_{\lambda,0} = U_\lambda \cap X_0$ is representative in U_λ. The collection $\mathcal{U}_* = \{(U_\lambda, U_{\lambda,0})\}$ is a poset under the subset relation, so it can be regarded as a category with inclusions as morphisms.

Theorem 2.3 (Van Kampen, Brown)**.** *Let (X, X_0) and $\mathcal{U}_*$ be given as above. Then*

$$\Pi_1(X, X_0) \cong \operatorname*{colim}_{\mathcal{U}_*} \Pi_1(U_\lambda, U_{\lambda,0}).$$

Presently we give a sketch of the proof and comment on the hypothesis of the theorem along the way. The full proof consists of fairly natural steps, but it is a bit lengthy and at points technical, so we defer it to the appendix.

The functor Π_1 gives a diagram $\mathcal{U}_* \to \mathbf{Grpd}$ which we denote $\Pi_1|_{\mathcal{U}_*}$. We think of the groupoid $\Pi_1(X, X_0)$ as a constant functor. The theorem says that $\Pi_1(X, X_0) \cong \operatorname{colim} \Pi_1|_{\mathcal{U}_*}$. Subspace inclusion of $(U_\lambda, U_{\lambda,0})$ into (X, X_0) makes $\Pi_1(X, X_0)$ a cocone under $\Pi_1|_{\mathcal{U}_*}$.

The main point of the proof is to show that the cocone is universal. This amounts to showing that we have the following commutative diagram

$$\begin{array}{ccc} \Pi_1|_{\mathcal{U}_*} & \Longrightarrow & \Pi_1(X, X_0) \\ & \searrow\!\!\!\searrow & \downarrow{\scriptstyle \exists!} \\ & & \mathcal{G} \end{array}$$

Our definition of the functor $\Pi_1(X, X_0) \to \mathcal{G}$ will be guided by the commutativity of the diagram. It is also the reason why the functor is unique. Defining the functor on the morphisms of $\Pi_1(X, X_0)$ is the hard bit. This requires subdividing the morphism so that each piece lies entirely in some U_λ. Then we deform via a homotopy the endpoints of each such piece so that they are in $U_{\lambda,0}$ and use $\Pi_1|_{\mathcal{U}_*} \Rightarrow \mathcal{G}$. This is why we need $U_{\lambda,0}$ to be representative in U_λ.

We need to verify that the functor is well-defined on morphisms. This is done in two steps. Given two paths representing the same morphism, each with its own subdivision, we create a common refinement and join them up using a homotopy that respects the refined subdivision. Then we deform the homotopy connecting the two subdivisions to a better homotopy that allows us to use the commutativity of the diagram, which shows that the functor is well-defined.

The step where we deform the homotopy is why the open cover $\mathcal{U}$ needs to be finite. The van Kampen theorem is actually also true for infinite $\mathcal{U}$ and X_0 only has to be representative in three-fold intersections of open sets in $\mathcal{U}$ if we reformulate the statement using coequalizers (a special type of colimit)[4]. The argument there is similar to the one presented here, except that the notion of Lebesgue covering dimension is needed.

2.3 Fundamental group of the circle

The van Kampen theorem as it is produces a groupoid with multiple base points. To obtain the fundamental group, we perform a second retraction [3, p. 245] to reduce the number of base points down to one. From there, we show that $B\mathbb{Z}$ is the fundamental group by verifying universal properties.

Proposition 2.4. *Suppose that $f : \mathcal{C} \to \mathcal{D}$ is a functor injective on objects. Let $i : \mathcal{C}' \to \mathcal{C}$ be a subcategory such that i is full and essentially surjective. Write the restriction of the functor f to $\mathcal{C}'$ as f'. Then there is a subcategory $j : \mathcal{D}' \to \mathcal{D}$ with deformation retractions $r : \mathcal{C} \to \mathcal{C}'$ and $r' : \mathcal{D} \to \mathcal{D}'$ making the following diagram a pushout.*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{r} & \mathcal{C}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathcal{D} & \xrightarrow{r'} & \mathcal{D}' \end{array} \tag{*}$$

Proof. We define the objects of $\mathcal{D}'$ to be $\mathrm{Ob}\,\mathcal{D}' = f(\mathrm{Ob}\,\mathcal{C}') \cup (\mathrm{Ob}\,\mathcal{D} \setminus f(\mathrm{Ob}\,\mathcal{C}))$. For each pair $d, d' \in \mathrm{Ob}\,\mathcal{D}'$, we define $\mathcal{D}'(d,d') = \mathcal{D}(d,d')$. The composition of morphisms in $\mathcal{D}'$ is inherited from $\mathcal{D}$ and this makes $\mathcal{D}'$ a subcategory of $\mathcal{D}$.

By construction, the inclusion functor $j : \mathcal{D}' \to \mathcal{D}$ is fully faithful. The functor j is also essentially surjective by definition. In principle, we can obtain a deformation retraction just from the definition of $\mathcal{D}'$. However, this is not enough for our purpose because not any old deformation retraction will satisfy the pushout square. Instead, we construct the deformation retraction r' by simultaneously exhibiting a natural isomorphism μ.

The corollary of proposition 1.2 gives a deformation retraction $r : \mathcal{C} \to \mathcal{C}'$ and a natural isomorphism $\eta : \mathrm{id}_{\mathcal{C}} \cong ir$. Suppose that $p : c_0 \to c_0'$ is a morphism in $\mathcal{C}$ with $\eta_{c_0} : c_0 \to c$ and $\eta_{c_0'} : c_0' \to c'$ being the components of η at c_0 and c_0'.

We define μ by the following formula

$$\mu_{d_0} = \begin{cases} f\eta_{c_0} : fc_0 \to fc & \text{if } d_0 = fc_0 \text{ where } c_0 \in \mathrm{Ob}\,\mathcal{C}; \\ \mathrm{id}_{d_0} : d_0 \to d_0 & \text{if } d_0 \in \mathrm{Ob}\,\mathcal{D} \setminus f(\mathrm{Ob}\,\mathcal{C}). \end{cases}$$

The target of each component of μ is of the form jd for some object d in $\mathcal{D}'$. By the proof of proposition 1.2, there is a functor r' such that $\mu : \mathrm{id}_{\mathcal{D}} \cong jr'$. By the remark following the corollary of proposition 1.2, if c_0 is an object in $\mathcal{C}'$, then $c = c_0$ and $\eta_{c_0} = \mathrm{id}_{c_0}$. Since $\eta_{c_0} = \mathrm{id}_{c_0}$ when c_0 is an object of $\mathcal{C}'$, we have $\mathrm{id}_{\mathcal{D}} \cong jr' \operatorname{rel} \mathcal{D}'$, so r' is indeed a deformation retraction.

With the functors r and r' we just constructed, we have the commutative diagram (*). Indeed, if $\eta_{c_0} : c_0 \to c$, then we have

$$f'rc_0 = f'c = fc = r'fc_0,$$

showing that $f'r$ and $r'f$ agree on objects. Suppose now that $p : c_0 \to c_0'$ is a morphism in $\mathcal{C}$. Recall that in proposition 1.2, the morphisms rp and $r'fp$ are defined so that the following diagram commutes

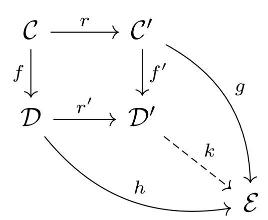
$$\begin{array}{ccc} c_0 & \xrightarrow{\eta_{c_0}} & c \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle irp} \\ c_0' & \xrightarrow{\eta_{c_0'}} & c' \end{array} \quad \overset{f}{\leadsto} \quad \begin{array}{ccc} fc_0 & \xrightarrow{\mu_{fc_0}} & fc \\ {\scriptstyle fp}\downarrow & & \downarrow{\scriptstyle firp} \\ fc_0' & \xrightarrow{\mu_{fc_0'}} & fc' \end{array}$$

Since the morphisms $r'fp$ and $f'rp$ both make the right-hand square commute and the morphisms μ_{fc_0} and $\mu_{fc_0'}$ are both isomorphisms, we get $r'fp = f'rp$, which shows $r'f = f'r$.

We just need to verify that if we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{r} & \mathcal{C}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{D} & \xrightarrow{h} & \mathcal{E} \end{array}$$

then there is a unique functor $k : \mathcal{D}' \to \mathcal{E}$ making the diagram below commute.



Such a functor k must satisfy the equation $kr' = h$. Thus, $k = kr'j = hj$. This means the functor k is unique and that we must take $k = hj$ as the definition.

We first compute $h\mu_d$ where $d \in \operatorname{Ob} \mathcal{D}$. If d is not in $f(\operatorname{Ob} C)$, then $h\mu_d = h\mathrm{id}_d = \mathrm{id}_{hd}$. If $d = fc_0$ and $\eta_{c_0} : c_0 \to c$ is the component of η at c_0, then the commutative square

$$\begin{array}{ccc} c_0 & \xrightarrow{\eta_{c_0}} & c \\ {\scriptstyle \eta_{c_0}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_c} \\ c & \xrightarrow{\eta_c = \mathrm{id}_c} & c \end{array}$$

shows $r\eta_{c_0} = \mathrm{id}_c$. Therefore, $h\mu_d = hf\eta_{c_0} = gr\eta_{c_0} = \mathrm{id}_{gc}$. The important point is that either case, $h\mu_d$ is an identity morphism.

We now show that $h = hjr'$. Let d_0 be an object of $\mathcal{D}$ and $\mu_{d_0} : d_0 \cong d = jr'd_0$. Applying h to both side of μ_{d_0} turns it into an identity morphism, so h and hjr' agree on objects. If $q : d_0 \to d_0'$, then

$$hjr's = h(\mu_{d_0'} s \mu_{d_0}^{-1}) = h\mu_{d_0'} \circ hs \circ h(\mu_{d_0}^{-1}) = hs.$$

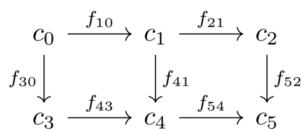
This shows that $h = hjr' = kr'$.

Now, we only need to show that $g = kf'$. Note that the functor f' satisfies the equation $jf' = fi$. The calculation

$$kf' = hjf' = hfi = gri = g.$$

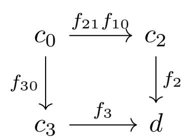
shows that the diagram (*) is a pushout. $\square$

Proposition 2.5. *Pushouts can be composed. More precisely, consider the following commutative diagram in a category $\mathcal{C}$ where the left and right squares are both pushouts.*

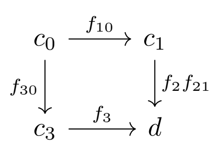


Then the outer rectangle is a pushout.

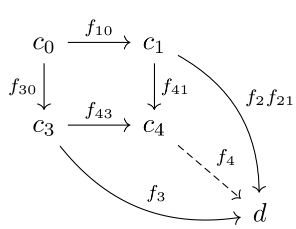
Proof. Let d be another object in $\mathcal{C}$ making the square below commute.



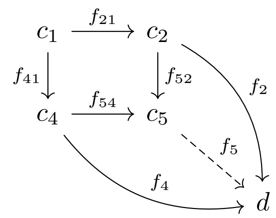
Considering $f_2 f_{21}$ as a map from c_1, we have the commutative square



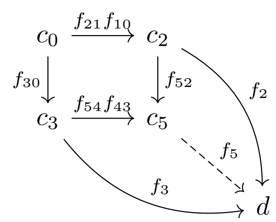
Since the left square is a pushout, we have an induced map $f_4 : c_4 \to d$ making the diagram below commute.



The right square is a pushout, so we have an induced map f_5 making the diagram below commutative.



Using the map f_5, we have a commutative diagram

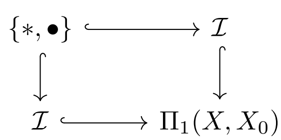


Since the left and right squares are both pushouts, the maps f_4 and f_5 are unique. In particular, the outer rectangle is a pushout. $\square$

Theorem 2.6. *$\pi_1(S^1, 1) \cong B\mathbb{Z}$.*

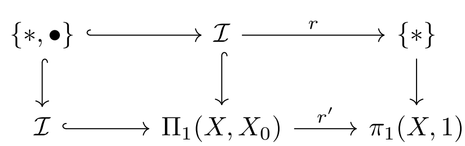
Proof. Let $X = S^1 \subset \mathbb{C}$. Consider the base points $X_0 = \{-1, 1\}$ and the open cover $\mathcal{U} = \{U_1, U_2, U_{12}, X\}$ where $U_1 = S^1 \setminus \{i\}, U_2 = S^1 \setminus \{-i\}$. Let $U_{1,0} = U_1 \cap X_0$, $U_{2,0} = U_2 \cap X_0$ and $U_{12,0} = U_{12} \cap X_0$. In example 2.1, we calculated that $\Pi_1(U_1, U_{1,0}) = \Pi_1(U_2, U_{2,0}) = \mathcal{I}$. Then we found in example 2.2 that $\Pi_1(U_{12}, U_{12,0}) = \{*, \bullet\}$ is the discrete groupoid of two elements.

By the van Kampen theorem, we have a pushout diagram

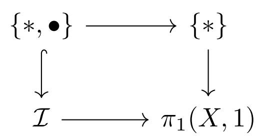


Since the functor $\mathcal{I} \hookrightarrow \Pi_1(X, X_0)$ is induced by inclusion of topological spaces, it is clearly injective on objects. Consider the discrete subcategory $\{*\}$ of $\mathcal{I}$. As the morphism $s : * \to \bullet$ is an isomorphism and we have a unique morphism from $*$ to itself in $\mathcal{I}$, the inclusion $\{*\} \hookrightarrow \mathcal{I}$ is fully faithful and essentially surjective.

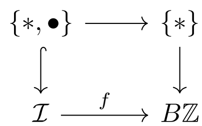
The fundamental groupoid of X based at 1 is $\pi_1(X, 1) = \Pi_1(X, \{1\})$. We have the following commutative diagram, where the right square is a pushout by proposition 2.4



Using proposition 2.5 we can glue the two pushout squares to obtain the pushout



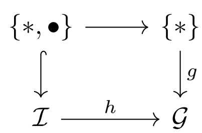
We show that we also have a pushout



As colimits in general are unique up to isomorphisms, this proves the theorem.

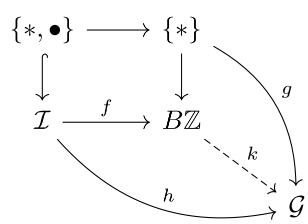
Let $s : * \to \bullet$ be the unique morphism in $\mathrm{Hom}_{\mathcal{I}}(*, \bullet)$. Defining a functor from $\mathcal{I}$ to a category is the same as picking an invertible morphism (i.e. isomorphism) in that category and sending s to it. Let us define the functor $f : \mathcal{I} \to B\mathbb{Z}$ by $f(s) = 1$. The square commutes by definition of f.

Let $\mathcal{G}$ be a groupoid. Suppose that we have the commutative square



By the commutativity of the square, h must map both objects of $\mathcal{I}$ to a common object x in $\mathcal{G}$. This means the unique object of $B\mathbb{Z}$ must be sent to x. Since $\mathrm{Hom}_{\mathcal{G}}(x, x)$ is a group, a functor from $B\mathbb{Z}$ is now exactly the same as a group homomorphism from $\mathbb{Z} \to \mathrm{Hom}_{\mathcal{G}}(x, x)$.

The commutativity of the diagram



forces us to define $k(1) = k(f(s)) = h(s)$. A homomorphism $\mathbb{Z} \to \mathrm{Hom}_{\mathcal{G}}(x, x)$ is uniquely determined by the image of $1 \in \mathbb{Z}$. Therefore, the functor k making the diagram commute exists and is unique. This concludes the proof. □

A Proof of the van Kampen theorem

The key to proving the van Kampen theorem is the idea of subdivisions. Given a map α from a compact metric space to a space X and an open cover $\mathcal{U} = \{U_\lambda\}$ of X, we often want to subdivide the domain into smaller pieces so that the image of each piece under α lies entirely in some open set in $\mathcal{U}$. The Lebesgue number lemma guarantees this is possible if the pieces are chosen sufficiently small. We state the lemma without proof since it is a standard fact [3, p. 91].

Lemma A.1 (Lebesgue number lemma)**.** *Let $\mathcal{U}$ be an open covering of a compact metric space (X, d). There is a $\delta > 0$ such that for each subset of X having a diameter less than δ, there exists an element of $\mathcal{U}$ containing it. This number δ is called a Lebesgue number for $\mathcal{U}$.*

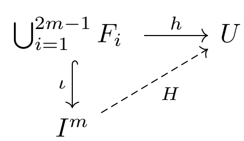
How one subdivides the domain depends on the problem at hand. For instance, the excision theorem for homology can be proved by barycentric subdivision of simplices [7, p. 119]. For an absolute version of the groupoid van Kampen theorem, we can choose the subdivision almost naively [9]. One may follow this swift and elegant proof by a categorical retraction lemma to deduce a van Kampen theorem for the fundamental groupoid relative to a set of base points [3, pp.240-245].

We shall, however, take a slightly different route. Upon closer examination of the proof of the absolute version of the groupoid van Kampen theorem, one sees that if we are slightly more careful with the subdivision, we can deduce a more general version relative to a set of base point without performing a categorical retraction. This has the advantage of being more amenable to generalization to higher dimensions [5, p. 30]. The geometric modifications are fairly natural, so we first set up the necessary tools for the change.

The standard m-dimensional cube I^m has $2m$ faces, each homeomorphic to I^{m-1}. Let $F_1, \dots, F_{2m}$ be the faces of I^m and the labelling chosen so that $F_1 = \{(x_1, \dots, x_{n-1}, 0) : 0 \le x_i \le 1\}$ and $F_{2m} = \{(x_1, \dots, x_{n-1}, 1) : 0 \le x_i \le 1\}$. In other words, F_1 is the "bottom face" of I^m and F_{2m} is the "top face" of I^m.

We first show a result that allows us to construct homotopies that satisfy certain desired conditions.

Lemma A.2. *Let $F_1, \dots, F_{2m}$ be the faces of I^m, where F_1 is the bottom face and F_{2m} is the top face. Suppose that U is a topological space and $h : F_1 \cup \dots \cup F_{2m-1} \to U$ is continuous. Then there is a continuous map $H : I^m \to U$ making the following diagram commute.*



In particular, $H : h|_{F_1} \simeq H|_{F_{2m}}$.

Proof. Let $P = (\frac{1}{2}, \dots, \frac{1}{2}, 2) \in \mathbb{R}^m$. We can define a continuous surjective map $p : I^m \to \bigcup_{i=1}^{2m-1} F_i$ by projecting from the point P. For each point $x \in I^m$, there is a unique ray passing through the points P and x. We define $p(x)$ to be the unique intersection of this ray and $\bigcup_{i=1}^{2m-1} F_i$.

Then we may define $H : I^m \to U$ by the formula $H(x) = h(p(x))$. If $x \in \bigcup_{i=1}^{2m-1} F_i$, then $H(\iota(x)) = h(p(\iota(x))) = h(x)$ because $p(\iota(x)) = x$. $\square$

Lemma A.2 also allows us to construct a "filling" of any space that is homeomorphic to I^m by composing with an appropriate homeomorphism. This is helpful because we will apply the lemma to cubes that are translated and scaled.

Given a map from a cube and a subdivision, we can obtain a new map that behaves in a nicer way on the subdivision structure via a homotopy. We make the idea precise in the next lemma. The proof involves applying lemma A.2 inductively. Let us fix some set-up for the remaining part of the appendix.

Suppose that $\mathcal{U} = \{U_\lambda\}$ is a finite open cover for X such that the intersection of finitely many open sets of $\mathcal{U}$ is again in $\mathcal{U}$. Let $X_0 \subset X$ be such that $U_{\lambda,0} = U_\lambda \cap X_0$ is representative in U_λ and form the collection $\mathcal{U}_* = \{(U_\lambda, U_{\lambda,0})\}$.

Let the cube I^m be subdivided by planes parallel to the coordinate hyperplanes $x_i = 0$ with $1 \leq i \leq m$, into closed cubes c_l. For each integer $0 \leq n \leq m$, we will denote by I_n the set of all n-dimensional faces of cubes occurring in the subdivision of I^m. For example, $v \in I_0$ if it is the vertex (homeomorphic to I^0) of some cube c_l of the subdivision. An element $e \in I_1$ is the edge (homeomorphic to I^1) of some cube c_l of the subdivision. Similarly for other I_n. In particular, the set of subcubes c_l is exactly I_m. We call the set I_n the n-skeleton [1] of I^m.

Now we state and prove the following lemma [2, p. 219].

Lemma A.3. *Let $\alpha : I^m \to X$ be continuous. Let the cube I^m be subdivided by planes parallel to the coordinate hyperplanes $x_i = 0$ with $1 \leq i \leq m$, into closed cubes c_l such that for each c_l, there is an open set U_λ of $\mathcal{U}$ with $\alpha(c_l) \subset U_\lambda$.*

Given the conditions we imposed, there is a map $\theta : I^m \to X$ that maps the vertices of each cube c_l into X_0 with a homotopy $H : \alpha \simeq \theta$ satisfying the following two conditions:

i) if $C \in I_n$ with $\alpha(C) \subset U_\lambda$, then $H(C \times I) \subset U_\lambda$;

ii) if F_0 is a face of some $C \in I_n$ and $\alpha(F_0) \subset X_0$ already, then H is the constant homotopy on F_0. In particular, $H(F_0 \times I) \subset X_0$.

Proof. Constructing a homotopy $H : \alpha \simeq \theta$ is the same as defining a map from $I^m \times I \to X$ that restricts to α on the bottom face and θ on the top face. The strategy of the proof is to define H inductively on the n-skeleton of I^m.

Let $v \in I_0$ be a vertex of the subdivision. Since the cover $\mathcal{U}$ is finite, we consider the intersection U_λ of all those open sets that contain $\alpha(v)$. By assumption, $\mathcal{U}$ is closed under finite intersections, so $U_\lambda \in \mathcal{U}$. If $\alpha(v) \in U_{\lambda,0}$, then we define $H|_{\{v\}\times I}$ to be the constant path at $\alpha(v)$ and $\theta(v) = \alpha(v)$. If $\alpha(v) \notin U_{\lambda,0}$, there is an $x_0 \in U_{\lambda,0}$ together with a path from $\alpha(v)$ to x_0 that lies entirely in U_λ. Define $H|_{\{v\}\times I}$ to be this path and $\theta(x) = x_0$.

We emphasize the values of θ at this stage because θ sends the vertices in the subdivision into X_0 as we wanted. What we do later will not affect it. Notice that by construction, condition i) is satisfied in the case $n = 0$, condition ii) is satisfied in the case $n = 1$.

Suppose that for each $0 \leq k \leq n-1$ and $F \in I_k$, we have defined $H|_{F\times I}$ so that condition i) is satisfied for $0 \leq k \leq n-1$ and ii) is satisfied for $1 \leq k \leq n$.

[1] An n-skeleton is technically the union of elements in what we call an n-skeleton; we choose this name because the subdivision makes the cube I^m into a CW-complex, but we do not need any facts about CW-complexes in the proof of lemma A.3.

Let $C \in I_n$ and $F_1, \ldots, F_{2n}$ be the $2n$ faces of C. As C is part of some cube c_l, $\alpha(C)$ is contained entirely in some open set of $\mathcal{U}$. We can again take the intersection U_λ of all open sets containing $\alpha(C)$ and $U_\lambda \in \mathcal{U}$.

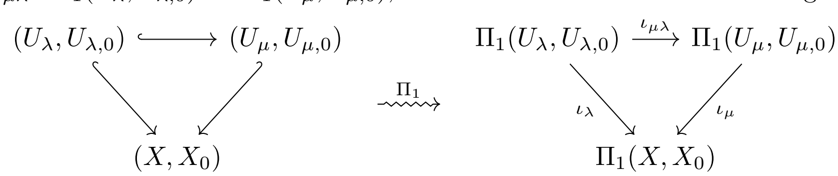
We identify $C \times I$ with $I^n \times I$ with C being the bottom face and $F_1 \times I, \ldots, F_{2n} \times I$ being the side faces. Similar to the 0-skeleton, if $\alpha(C) \subset U_{\lambda,0}$, then the easiest way to satisfy condition ii) for the $(n+1)$th case is to define $H|_{C \times I}$ to be the constant homotopy. The function $H|_{C\times I}$ is indeed continuous because each F_i lies in $U_{\lambda,0}$ and $H|_{F_i \times I}$ is a constant homotopy by induction. This function $H|_{C\times I}$ also satisfies condition i) for the nth case.

The other possibility is $\alpha(C) \not\subset U_{\lambda,0}$. The constant homotopy construction as shown above applied at the step going from I_{n-2} to I_{n-1} shows that condition ii) in the nth case is already met by the homotopies $H|_{F_1\times I}, \ldots, H|_{F_1 \times I}$ defined by induction. Lemma A.2 gives a homotopy $H|_{C\times I}$ that restricts to $H|_{F_1\times I}, \ldots, H|_{F_{2n}\times I}$ and $\alpha|_C$. The homotopy $H|_{C\times I}$ satisfies condition i) in the nth case. This completes the inductive construction. $\square$

Now we are ready to give a proof of the van Kampen theorem.

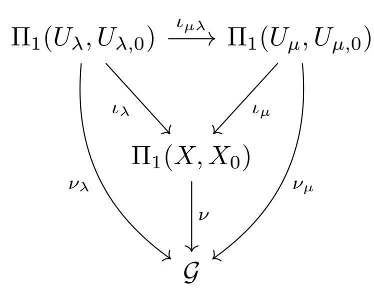
Proof of theorem 2.3. We show that $\Pi_1(X, X_0)$ satisfies the universal property of the colimit in question.

The standard inclusion map $(U_\lambda, U_{\lambda,0}) \hookrightarrow (X, X_0)$ induces a map $\iota_\lambda : \Pi_1(U_\lambda, U_{\lambda,0}) \hookrightarrow \Pi_1(X, X_0)$. If the induced map of $(U_\lambda, U_{\lambda,0}) \hookrightarrow (U_\mu, U_{\mu,0})$ is $\iota_{\mu\lambda} : \Pi_1(U_\lambda, U_{\lambda,0}) \hookrightarrow \Pi_1(U_\mu, U_{\mu,0})$, then we have the commutative diagrams



This shows that the system (ι_λ) forms a natural transformation from the functor $\Pi_1 : \mathcal{U}_* \to \mathbf{Grpd}$ to the constant functor $\Pi_1(X, X_0) : \mathcal{U}_* \to \mathbf{Grpd}$.

Suppose $\mathcal{G}$ is a groupoid that forms a cocone under the functor $\Pi_1 : \mathcal{U}_* \to \mathbf{Grpd}$. Let the natural transformation from Π_1 on $\mathcal{U}_*$ to $\mathcal{G}$ be given by (ν_λ). We need to show that there is a unique functor $\nu : \Pi_1(X, X_0) \to \mathcal{G}$ that makes the following diagram commute.



for each pair of λ and μ.

Defining ν on objects is quite straightforward. If $x \in X$, then $x \in U_\lambda$ for some λ. Due to the diagram above, the functor ν has to satisfy $\nu \circ \iota_\lambda$, so we have to define $\nu(x) = \nu_\lambda(x)$. We need to show that ν is well-defined on objects. Indeed, suppose we have $x \in U_\mu$. The open set $U_\kappa = U_\lambda \cap U_\mu$ is in $\mathcal{U}$. By the commutativity of the diagram below

$$\begin{array}{ccccc} & & \Pi_1(U_\kappa, U_{\kappa,0}) & & \\ & \swarrow^{\iota_{\lambda\kappa}} & & \searrow^{\iota_{\mu\kappa}} & \\ \Pi_1(U_\lambda, U_{\lambda,0}) & & & & \Pi_1(U_\mu, U_{\mu,0}) \\ & \searrow_{\nu_\lambda} & & \swarrow_{\nu_\mu} & \\ & & \mathcal{G} & & \end{array}$$

we have $\nu_\lambda(x) = \nu_\mu(x)$. This shows $\nu(x)$ does not depend on the cover x is in.

Let $\alpha : x \to y$ be a path in X joining two points $x, y \in X_0$. We would like to subdivide I as

$$0 = t_0 < t_1 < \cdots < t_l = 1.$$

The Lebesgue number lemma tells us that we can choose t_i and l suitably, so that each $\alpha([t_i, t_{i+1}]) \subset U_\lambda$ for some λ. Let us write U_i instead to emphasize that this open set contains $\alpha([t_i, t_{i+1}])$. Similarly, we write $U_{i,0}$ for $U_{\lambda,0}$ and ν_i for ν_λ. We cannot use the cocone condition yet to determine where $\alpha|_{[t_i,t_{i+1}]}$ must be sent because $\alpha(t_i)$ and $\alpha(t_{i+1})$ are most likely not in $U_{i,0}$.

This is where lemma A.3 helps us. Using the $m = 1$ case of lemma A.3, we construct a path θ from x to y with both $\theta(t_i), \theta(t_{i+1}) \in U_{i,0}$, together with a homotopy $H : \alpha \simeq \theta$ that is constant on $\alpha(t_i)$ if $\alpha(t_i) \in U_{i,0}$.

We define $\nu(\alpha) = \nu_{n-1}(\theta|_{[t_{n-1},t_n]}) \ldots \nu_0(\theta|_{[t_0,t_1]})$, omitting the $\cdot$ in concatenation of paths for brevity. Again, we need to show that ν is well-defined on morphisms. If we have a homotopy $K : \alpha \simeq \alpha' \operatorname{rel} \{x, y\}$, then we want to show that $\nu(\alpha) = \nu(\alpha')$.

Our procedure produces $\nu(\alpha')$ by subdividing the interval I into

$$0 = t'_0 < t'_1 < \cdots < t'_{l'} = 1.$$

so that each $\alpha'([t'_j, t'_{j+1}])$ is contained in some U'_j. Lemma A.3 gives us a homotopy $H' : \alpha' \simeq \theta'$ that is constant on $\alpha'(t'_j)$ if $\alpha(t'_j) \in U'_{j,0}$.

Let R be the reverse homotopy of H. By concatenating homotopies as in definition 2.2, we obtain a homotopy $G = H' \cdot K \cdot R : \theta \simeq \theta'$. We would like to modify the homotopy G using the $m = 2$ case of lemma A.3. Then we can use ν_i and ν_j to show that $\nu(\alpha) = \nu(\alpha')$.

We begin by subdividing I^2 parallel to the coordinate axes. By the Lebesgue number lemma, if we subdivide $I^2 = I \times I$ into squares with sufficiently small diameters, G must map each square entirely into some open set in $\mathcal{U}$. Let the first component of I^2 be subdivided into

$$0 = \tau_0 < \tau_1 < \cdots < \tau_n = 1$$

with each t_i and t'_j occurring in the subdivision. Let the second component of I^2 be subdivided into

$$0 = \sigma_0 < \sigma_1 < \cdots < \sigma_r = 1.$$

We have $G([\tau_p, \tau_{p+1}] \times [\sigma_q, \sigma_{q+1}]) \subset U_{pq}$, where $U_{pq} \in \mathcal{U}$. Lemma A.3 gives us a homotopy $L : G \simeq \overline{G}$ with the property that $\overline{G}([\tau_p, \tau_{p+1}] \times [\sigma_q, \sigma_{q+1}]) \subset U_{pq}$ and $\overline{G}(\tau_p, \sigma_q) \in U_{pq,0}$ for every vertex (τ_p, σ_q) of the subdivision of I^2.

Let $\overline{\theta}$ the path defined by $\overline{G}|_{I\times\{0\}}$ and $\overline{\theta'}$ the path defined by $\overline{G}|_{I\times\{1\}}$.

Consider $\theta|_{[t_i,t_{i+1}]}$. The interval $[t_i, t_{i+1}]$ is already subdivided into

$$t_i = \tau_{p_i} < \tau_{p_i+1} < \cdots < \tau_{p_{i+1}} = t_{i+1}.$$

The first condition of lemma A.3 tells us that each $L([\tau_{p_i+k}, \tau_{p_i+k+1}] \times \{0\} \times I) \subset U_i$, so $L([t_i, t_{i+1}] \times \{0\} \times I) \subset U_i$. The homotopy

$$L|_{[t_i,t_{i+1}]\times\{0\}\times I} : \theta|_{[t_i,t_{i+1}]} \simeq \overline{\theta}|_{[t_i,t_{i+1}]}$$

lies entirely in U_i with $L(\{t_i\} \times \{0\} \times I), L(\{t_{i+1}\} \times \{0\} \times I) \subset U_{i,0}$. This gives the equality $\nu_i(\theta|_{[t_i,t_{i+1}]}) = \nu_i(\overline{\theta}|_{[t_i,t_{i+1}]})$. Moreover,

$$\overline{\theta}|_{[t_i,t_{i+1}]} = \overline{\theta}|_{[\tau_{p_{i+1}},\tau_{p_{i+1}-1}]} \ldots \overline{\theta}|_{[\tau_{p_i},\tau_{p_i+1}]}.$$

Let the intersection of U_i and $U_{(p_i+k)0}$ be U_k. The commutative diagram

$$\begin{array}{ccccc} & & \Pi_1(U_k, U_{k,0}) & & \\ & \swarrow^{\iota_{(p_i+k)0,k}} & & \searrow^{\iota_{ik}} & \\ \Pi_1(U_{(p_i+k)0}, U_{(p_i+k)0,0}) & & & & \Pi_1(U_i, U_{i,0}) \\ & \searrow_{\nu_{(p_i+k)0}} & & \swarrow_{\nu_i} & \\ & & \mathcal{G} & & \end{array}$$

shows

$$\begin{aligned} \nu_i(\overline{\theta}|_{[t_i,t_{i+1}]}) &= \nu_i(\overline{\theta}|_{[\tau_{p_{i+1}-1},\tau_{p_{i+1}}]}) \ldots \nu_i(\overline{\theta}|_{[\tau_{p_{i+1}-1},\tau_{p_{i+1}}]}) \\ &= \nu_{p_{i+1}0}(\overline{\theta}|_{[\tau_{p_{i+1}-1},\tau_{p_{i+1}}]}) \ldots \nu_{p_i0}(\overline{\theta}|_{[\tau_{p_i},\tau_{p_i+1}]}). \end{aligned}$$

Therefore,

$$\nu(\alpha) = \nu_{l-1}(\theta|_{[t_{l-1},t_l]}) \ldots \nu_0(\theta|_{[t_0,t_1]}) = \nu_{(n-1)0}(\overline{\theta}|_{[\tau_{n-1},\tau_n]}) \ldots \nu_{00}(\overline{\theta}|_{[\tau_0,\tau_1]}).$$

Similarly,

$$\nu(\alpha') = \nu_{l'-1}(\theta'|_{[t'_{l'-1},t'_{l'}]}) \ldots \nu_0(\theta'|_{[t'_0,t'_1]}) = \nu_{(n-1)(r-1)}(\overline{\theta'}|_{[\tau_{n-1},\tau_n]}) \ldots \nu_{0(r-1)}(\overline{\theta'}|_{[\tau_0,\tau_1]}).$$

We are in a much better position now because the homotopy $\overline{G}$ is exactly what we need to show that

$$\nu_{(n-1)0}(\overline{\theta}|_{[\tau_{n-1},\tau_n]}) \ldots \nu_{00}(\overline{\theta}|_{[\tau_0,\tau_1]}) = \nu_{(n-1)(r-1)}(\overline{\theta'}|_{[\tau_{n-1},\tau_n]}) \ldots \nu_{0(r-1)}(\overline{\theta'}|_{[\tau_0,\tau_1]}).$$

Recall that we have $\overline{G}([\tau_p, \tau_{p+1}] \times [\sigma_q, \sigma_{q+1}]) \subset U_{pq}$. As all vertices of the square $[\tau_p, \tau_{p+1}] \times [\sigma_q, \sigma_{q+1}]$ are in $U_{pq,0}$, it makes sense to ask whether

$$\overline{G}|_{[\tau_p,\tau_{p+1}]\times\{\sigma_q\}} = \overline{G}|^{-1}_{\{\tau_{p+1}\}\times[\sigma_q,\sigma_{q+1}]} \overline{G}|_{[\tau_p,\tau_{p+1}]\times\{\sigma_{q+1}\}} \overline{G}|_{\{\tau_p\}\times[\sigma_q,\sigma_{q+1}]}$$

as morphisms in $\Pi_1(U_{pq}, U_{pq,0})$. The answer is yes and a homotopy can be given by modifying the projection map of lemma A.2 suitably. As ν_{pq} is a functor, we have an equality of "conjugate" morphisms in G

$$\nu_{pq}(\overline{G}|_{[\tau_p,\tau_{p+1}]\times\{\sigma_q\}}) = \nu_{pq}(\overline{G}|_{\{\tau_{p+1}\}\times[\sigma_q,\sigma_{q+1}]})^{-1}\nu_{pq}(\overline{G}|_{[\tau_p,\tau_{p+1}]\times\{\sigma_{q+1}\}})\nu_{pq}(\overline{G}|_{\{\tau_p\}\times[\sigma_q,\sigma_{q+1}]}).$$

Note that $G : \theta \simeq \theta' \operatorname{rel}\{x, y\}$. By the second condition in lemma A.3, we have $\overline{G} : \overline{\theta} \simeq \overline{\theta'} \operatorname{rel}\{x, y\}$. This means for any $[\sigma_q, \sigma_{q+1}]$, both $\overline{G}|_{\{0\}\times[\sigma_q,\sigma_{q+1}]}$ and $\overline{G}|_{\{1\}\times[\sigma_q,\sigma_{q+1}]}$ are constant paths. Thus, $\nu_{0q}(\overline{G}|_{\{0\}\times[\sigma_q,\sigma_{q+1}]})$ is the identity morphism on $\nu_{0q}(\overline{G}(0, \sigma_q))$.

By restricting to the intersection $U_{pq} \cap U_{p(q+1)}$ as we have done a few times already and conjugating morphisms, we get the following sequence of equalities.

$$\begin{aligned}
&\nu(\alpha)\\
=&\nu_{(n-1)0}(\overline{\theta}|_{[\tau_{n-1},\tau_n]}) \ldots \nu_{00}(\overline{\theta}|_{[\tau_0,\tau_1]})\\
=&\nu_{(n-1)0}(\overline{G}|_{[\tau_{n-1},\tau_n]\times\{\sigma_1\}}) \ldots \nu_{00}(\overline{G}|_{[\tau_0,\tau_1]\times\{\sigma_1\}})\\
=&\nu_{(n-1)1}(\overline{G}|_{[\tau_{n-1},\tau_n]\times\{\sigma_1\}}) \ldots \nu_{01}(\overline{G}|_{[\tau_0,\tau_1]\times\{\sigma_1\}})\\
&\vdots\\
=&\nu_{(n-1)(r-1)}(\overline{G}|_{[\tau_{n-1},\tau_n]\times\{\sigma_{r-1}\}}) \ldots \nu_{0(r-1)}(\overline{G}|_{[\tau_0,\tau_1]\times\{\sigma_{r-1}\}})\\
=&\nu_{(n-1)(r-1)}(\overline{\theta'}|_{[\tau_{n-1},\tau_n]}) \ldots \nu_{0(r-1)}(\overline{\theta'}|_{[\tau_0,\tau_1]})\\
=&\nu(\alpha').
\end{aligned}$$

This also showed that our definition of $\nu(\alpha)$ is independent of subdivision since that is the case when $\alpha' = \alpha$ and the homotopy K is the identity.

We have proved that ν is well-defined and commutes with the natural transformation (ν_λ) by construction. The fact that ν is a functor and is the unique functor $\Pi_1(X, X_0) \to \mathcal{G}$ commuting with the natural transformation (ν_λ) follows from subdividing the morphisms of $\Pi_1(X, X_0)$ and checking on $\Pi_1(U_\lambda, U_{\lambda,0})$. Hence, $\Pi_1(X, X_0) \cong \operatorname{colim}_{\mathcal{U}_*} \Pi_1(U_\lambda, U_{\lambda,0})$ as claimed. □

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