INTUITION FOR KAN EXTENSIONS, ENDS AND COENDS

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0. Introduction

Kan extensions, ends and coends are frequently occurring categorical tools. As with most mathematical concepts, there are a number of ways of thinking about them. Our goal here is to provide some intuition for these ideas by working out some examples. We will see that there is a good case for viewing them as generalizations of natural constructions for modules over a ring.

We will work without enrichment. This is so that we can get to the point quicker. A textbook account of the material here can be found in [1].

1. Ends and coends

1.1. **Balancing actions.** Our main examples will be representations, but we will use the language of modules. Let G be a group and k be a field. Given a (kG, kG)-bimodule M, we now give two ways to obtain a k-vector space. We can define the k-vector space $\mathrm{Bal}(M) = \{m \in M : gmg^{-1} = m, \forall g \in G\}$ by balancing the actions.[1] Dually, we can "cobalance" the

[1]This is nonstandard terminology; we are not referring to the notion of balanced module either.

actions on M by defining $\mathrm{Cobal}(M)$ to be the quotient of M by the k-vector subspace of M generated by the subset $\{m - gmg^{-1} : m \in M\}$.

Intuitively, the k-vector space $\mathrm{Bal}(M)$ is the largest vector subspace of M where the left action of G agrees with the right action of G. Similarly, the k-vector space $\mathrm{Cobal}(M)$ is the largest vector space quotient of M where the left action of G agrees with the right action of G. More precisely, if we regard M as a kG-module via the diagonal action, then $\mathrm{Bal}(M) = M^G$ is obtained by taking invariants and $\mathrm{Cobal}(M) = M_G$ is obtained by taking coinvariants.

Let us look at a typical example for each of these operation.

Given two left kG-modules M, N, the collection of k-linear maps $\mathrm{Hom}_k(M, N)$ is a (kG, kG)-bimodule in the usual way. To spell the details out, given $f \in \mathrm{Hom}_k(M, N)$ and $(h, g) \in G^{op} \times G$, we have $(gfh)(x) = gf(x)h$ for each $x \in M$. Then $\mathrm{Bal}(\mathrm{Hom}_k(M, N)) = \mathrm{Hom}_G(M, N)$ is simply the k-vector space of kG-linear maps.

For an example of cobalancing actions, consider a right kG-module M and a left kG-module N. We may tensor M and N over the ground field k to obtain a (kG, kG)-bimodule $M \otimes_k N$. The action of $(h, g) \in G^{op} \times G$ on $m \otimes n \in M \otimes_k N$ is given by $g(m \otimes n)h = mh \otimes gn$. Then $\mathrm{Cobal}(M \otimes_k N) = M \otimes_G N$ is the usual tensor product of M with N over kG.

We will revisit these examples after introducing Kan extensions.

1.2. **Recast in categorical language.** We would now like to generalize what we did in the previous section to arbitrary categories. The first step is to replace everything in sight with categories and functors in the usual way.

Let G be a group viewed as a one point category and k be a field. Then a left kG-module V is exactly a functor $G \to k\text{-}\mathbf{Mod}$. A right kG-module V is exactly a functor $G^{op} \to k\text{-}\mathbf{Mod}$. Finally, a (kG, kG)-bimodule M is a bifunctor $G^{op} \times G \to k\text{-}\mathbf{Mod}$.

How can we generalize the operations of balancing and cobalancing? Here is a natural first attempt.

The diagonal action of a bimodule $G^{op} \times G \to k\text{-}\mathbf{Mod}$ is obtained by precomposing the functor by the functor $\Delta : G \to G^{op} \times G$. In the first argument of Δ, we use the isomorphism of $G \cong G^{op}$ via $g \mapsto g^{-1}$. The second argument of Δ is simply the identity.

It can be easily checked that if $V : G \to k\text{-}\mathbf{Mod}$ is a left kG-module, then $\lim V \cong V^G$ and $\operatorname{colim} V \cong V_G$. For a (kG, kG)-bimodule M, then we have $\lim(M \circ \Delta) \cong \mathrm{Bal}(M)$ and $\operatorname{colim}(M \circ \Delta) \cong \mathrm{Cobal}(M)$.

However, this will not work when we replace the category G by an arbitrary category $\mathcal{C}$. In general, it is not true that $\mathcal{C} \cong \mathcal{C}^{op}$, so we cannot hope to pullback a general bifunctor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ to a functor $\mathcal{C}$.

Nevertheless, it is still possible to generalize the operation of balancing actions. This require the notion of wedges and ends, which we turn to now.

1.3. **Wedges and cowedges.** Let $\mathcal{F} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ be a bifunctor. Suppose that $w \in \mathcal{E}$ has morphisms $\eta_x : w \to \mathcal{F}(x, x)$. Then the object w together with the morphism $(\eta_x)_{x \in \mathcal{C}}$ is said to be a **wedge** over $\mathcal{F}$ if for every morphism $r : x \to y$, the square below commute.

$$\begin{array}{ccc} w & \xrightarrow{\eta_x} & \mathcal{F}(x, x) \\ {\scriptstyle \eta_y}\downarrow & & \downarrow{\scriptstyle r_*} \\ \mathcal{F}(y, y) & \xrightarrow{r^*} & \mathcal{F}(x, y) \end{array}$$

If $\mathcal{F} = M$ is a (kG, kG)-bimodule, then a wedge consists of a k-vector space V and a k-linear map $\eta : V \to M$. For each $g \in G$, the commutativity diagram is then

$$\begin{array}{ccc} V & \xrightarrow{\eta} & M \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle g_*} \\ M & \xrightarrow{g^*} & M \end{array}$$

The common composite is of course just η itself. Therefore, we need $g\eta(v) = \eta(v)g = \eta(v)$ for every $g \in G$ and $v \in V$. In other words, a wedge over the bimodule M is a k-linear map that factors through the k-vector subspace inclusion $i : \mathrm{Bal}(M) \to M$. Let us note that such a factorization is unique.

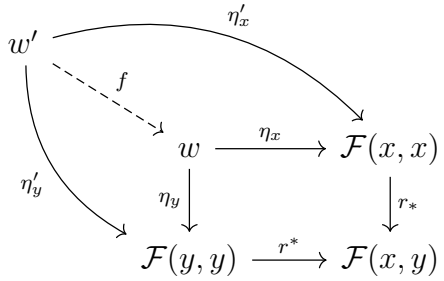
We can define a cowedge under the bifunctor $\mathcal{F}$ dually. Suppose $w \in \mathcal{E}$ has the morphism $\eta_x : \mathcal{F}(x, x) \to w$. Then the object w together with the morphism $(\eta_x)_{x \in \mathcal{C}}$ is said to be a **cowedge** under $\mathcal{F}$ if for every morphism $r : x \to y$, the square below commute.

$$\begin{array}{ccc} \mathcal{F}(x, y) & \xrightarrow{r_*} & \mathcal{F}(x, x) \\ {\scriptstyle r^*}\downarrow & & \downarrow{\scriptstyle \eta_x} \\ \mathcal{F}(y, y) & \xrightarrow{\eta_y} & w \end{array}$$

What is a cowedge under $\mathcal{F}$ when $\mathcal{F} = M$ is a (kG, kG)-bimodule? An analysis similar to the one carried out for wedges shows that a cowedge under M is a k-vector space V with a k-linear map $\eta : M \to V$ such that η factors through the quotient $q : M \to \mathrm{Cobal}(M)$. Again, such a factorization must be unique.

1.4. **Ends and coends.** Now we have the right tools to generalize the operation of balancing actions.

Let $\mathcal{F} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ be a bifunctor as before. A wedge $(w, (\eta_x)_{x \in \mathcal{C}})$ over $\mathcal{F}$ is said to an **end** of $\mathcal{F}$ if it is a universal wedge. This means whenever $(w', (\eta'_x)_{\mathcal{C}})$ is another wedge over $\mathcal{F}$, there is a unique morphism $f : w' \to w$ making the diagram below commute.



The usual argument shows that all ends of $\mathcal{F}$ are isomorphic up to unique isomorphisms. Thus, we can unambiguously talk about "the end" of $\mathcal{F}$ and will notate it as $\int_{c \in \mathcal{C}} \mathcal{F}$.

In the case that $\mathcal{F} = M$ is a (kG, kG)-bimodule, it is clear that the inclusion $i : \mathrm{Bal}(M) \to M$ is a wedge over M. Moreover, any wedge factors uniquely through i, so the end of M is $\int_{g \in G} M \cong \mathrm{Bal}(M)$.

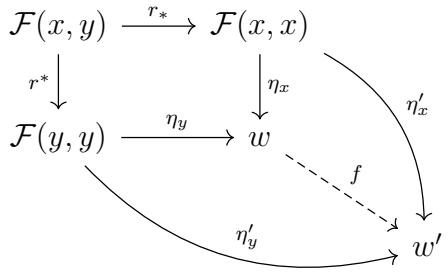
Here is a particularly enlightening example of an end.

Let $F, G : \mathcal{C} \to \mathcal{E}$ be functors. We have a bifunctor $\mathrm{Hom}_{\mathcal{E}}(F-, G-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$. What is the end $\int_{c \in \mathcal{C}} \mathrm{Hom}_{\mathcal{E}}(F-, G-)$? This is a nice exercise in understanding how ends work. We give the answer in the proposition below.

Proposition 1.1. *We have* $\int_{c\in\mathcal{C}} \mathrm{Hom}_{\mathcal{E}}(F-,G-) \cong \mathrm{Hom}_{\mathcal{E}^{\mathcal{C}}}(F,G)$*, where* $\mathcal{E}^{\mathcal{C}}$ *is the category of functors from* $\mathcal{C}$ *to* $\mathcal{E}$ *as usual.*

This is the unenriched analogue of the observation $\mathrm{Hom}_G(M,N) \cong \mathrm{Bal}(\mathrm{Hom}_k(M,N))$ that we made earlier!

Coends are defined dually. A cowedge $(w,(\eta_x)_{x\in\mathcal{C}})$ under $\mathcal{F}$ is said to a **coend** of $\mathcal{F}$ if it is a universal cowedge. This means whenever $(w',(\eta'_x)_{\mathcal{C}})$ is another cowedge under $\mathcal{F}$, there is a unique morphism $f : w \to w'$ making the diagram below commute.



The usual argument again shows that all coends of $\mathcal{F}$ are isomorphic up to unique isomorphisms. The notation for the coend of $\mathcal{F}$ is quite sensibly $\int^{c\in\mathcal{C}} \mathcal{F}$.

In the case that $\mathcal{F} = M$ is a (kG,kG)-bimodule, the quotient $q : M \to \mathrm{Cobal}(M)$ is a cowedge under M. Any cowedge factors uniquely through q, so the coend of M is $\int^{g\in G} M \cong \mathrm{Cobal}(M)$.

1.5. **Equalizer formula for ends.** We will present a formula for ends as an equalizer in this section. There is a dual formula for coends as a coequalizer. There are essentially no new ideas here; the only thing involved is linguistics.

Consider a bifunctor $\mathcal{F} : \mathcal{C}^{op}\times\mathcal{C} \to \mathcal{E}$. We have two maps $a,b : \prod_{x\in\mathcal{C}} \mathcal{F}(x,x) \to \prod_{f:x\to y} \mathcal{F}(x,y)$. The map a on the component of $\prod_{f:x\to y} \mathcal{F}(x,y)$ indexed by $f : x \to y$ is given by the composite

$$\prod_{x\in\mathcal{C}} \mathcal{F}(x,x) \xrightarrow{\pi_x} \mathcal{F}(x,x) \xrightarrow{f_*} \mathcal{F}(x,y).$$

The map b on the component of $\prod_{f:x\to y} \mathcal{F}(x,y)$ indexed by $f : x \to y$ is given by the composite

$$\prod_{x\in\mathcal{C}} \mathcal{F}(x,x) \xrightarrow{\pi_y} \mathcal{F}(y,y) \xrightarrow{f^*} \mathcal{F}(x,y).$$

Given a map $\eta : w \to \prod_{x\in\mathcal{C}} \mathcal{F}(x,x)$ in $\mathcal{E}$, we can define the morphism η_x by postcomposing η with $\pi_x : \prod_{x\in\mathcal{C}} \mathcal{F}(x,x) \to \mathcal{F}(x,x)$. The condition that $(\eta_x)_{x\in\mathcal{C}}$ makes w a wedge over $\mathcal{F}$ is equivalent to $a\eta = b\eta$. This can be verified by projecting to each component of $\prod_{f:x\to y} \mathcal{F}(x,y)$. From this we deduce at once the following.

Proposition 1.2. *The end of* $\mathcal{F} : \mathcal{C}^{op}\times\mathcal{C} \to \mathcal{E}$ *is*

$$\int_{c\in\mathcal{C}} \mathcal{F} \cong \mathrm{eq}\left(\prod_{x\in\mathcal{C}} \mathcal{F}(x,x) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{f:x\to y} \mathcal{F}(x,y)\right).$$

A nice thing to try now is to prove proposition 1.1 using the formula we just gave.

The story is similar for coends. We have two maps $a, b : \coprod_{f:x\to y} \mathcal{F}(x,y) \to \coprod_{x\in\mathcal{C}} \mathcal{F}(x,x)$. The map a is defined by the composite

$$\mathcal{F}(x,y) \xrightarrow{f_*} \mathcal{F}(x,x) \xrightarrow{\iota_x} \coprod_{x\in\mathcal{C}} \mathcal{F}(x,x).$$

The map b is defined by the composite

$$\mathcal{F}(x,y) \xrightarrow{f^*} \mathcal{F}(y,y) \xrightarrow{\iota_y} \coprod_{x\in\mathcal{C}} \mathcal{F}(x,x).$$

We deduce as before the following dual formula for coends as coequalizers.

Proposition 1.3. *The coend of $\mathcal{F} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ is*

$$\int^{c\in\mathcal{C}} \mathcal{F} \cong \operatorname{coeq}\left(\coprod_{f:x\to y} \mathcal{F}(x,y) \underset{b}{\overset{a}{\rightrightarrows}} \coprod_{x\in\mathcal{C}} \mathcal{F}(x,x) \right).$$

2. Kan extensions

We begin this section by reviewing two familiar constructions for representations. Then we indicate how the notion of left and right Kan extensions generalize them to categories. Finally, we prove a formula for left Kan extensions in terms of coends. This formula will help us understand Kan extensions and coends at the same time because their relationship mirrors the situation of representations quite closely.

2.1. **Restriction, extension and coextension of scalars.** Suppose that $\alpha : H \to G$ a group homomorphism. Any kG-module M can then be regarded as a kH-module α^*M by restriction of scalars along α. The action of H on M is defined by $hm = \alpha(h)m$.

The restriction of scalar α^* is a functor kG-**Mod** $\to kH$-**Mod**. Recall that it has both a left adjoint $\alpha_!$ called extension of scalars along α and a right adjoint α_* called coextension of scalars along α. Note that in the language of representations, extension of scalars is called induction and coextension of scalars is called coinduction.

Recall that any adjunction has an equivalent description as a pair of universal properties. Before we review the familiar construction of $\alpha_!$ and α_*, let us first unravel what the universal properties these adjunctions encode.

If N is a left kH-module and M is a kG-module, then the natural isomorphism

$$\operatorname{Hom}_G(\alpha_!N, M) \cong \operatorname{Hom}_H(N, \alpha^*M)$$

factors as

$$\operatorname{Hom}_G(\alpha_!N, M) \xrightarrow{\alpha^*} \operatorname{Hom}_H(\alpha^*\alpha_!N, \alpha^*M) \xrightarrow{\eta^*} \operatorname{Hom}_H(N, \alpha^*M).$$

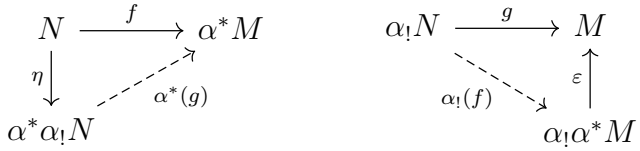
where $\eta : N \to \alpha^*\alpha_!N$ is the unit map. The inverse isomorphism factors as the composite

$$\operatorname{Hom}_H(N, \alpha^*M) \xrightarrow{\alpha_!} \operatorname{Hom}_H(\alpha_!N, \alpha_!\alpha^*M) \xrightarrow{\varepsilon_*} \operatorname{Hom}_G(\alpha_!N, M).$$

where $\varepsilon : \alpha_!\alpha^*M \to M$ is the counit map.

The universal property of this adjunction then says that for any kH-linear map $f : N \to \alpha^*M$ there is a unique kG-linear map $g : \alpha_!N \to M$ such that $f = \eta\alpha^*(g)$ and that for any

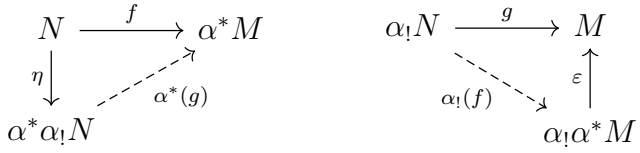
kG-linear map $g : \alpha_! N \to M$ there is a unique kH-linear map $f : N \to \alpha^* M$ such that $g = \alpha_!(f)\varepsilon$. In other words, we have the familiar commutative diagrams



There is a similar argument using the adjunction isomorphism

$$\mathrm{Hom}_H(\alpha^* M, N) \cong \mathrm{Hom}_G(M, \alpha_* N).$$

Let η and ε be the unit and counit of the adjunction. Then for any kH-linear map $f : \alpha^* M \to N$ there is a unique kG-linear map $g : M \to \alpha_* N$ such that $f = \eta\alpha^*(g)$ and that for any kG-linear map $g : \alpha_! N \to M$ there is a unique kH-linear map $f : N \to \alpha^* M$ such that $g = \alpha_!(f)\varepsilon$. In other words, we have the familiar commutative diagrams



Here is an explicit construction for the extension of scalars and coextension of scalars. Let us regard kG as (kG, kH)-bimodule, where G act on the left by multiplication in G and H act on the right by $lh = l\alpha(h)$ for each $l \in kG$ and $h \in H$. Then the extension of scalars $\alpha_! N$ is defined by $\alpha_! N = kG \otimes_H N$ with G acting on the left by $g(l \otimes n) = gl \otimes n$.

Now regard kG as (kH, kG)-bimodule, where G act on the right by multiplication in G and H act on the left by $hl = \alpha(h)l$ for each $l \in kG$ and $h \in H$. The coextension of scalars $\alpha_* N$ is defined by $\alpha_* N = \mathrm{Hom}_H(kG, N)$ with G acting on the left by $(gf)(n) = f(ng)$ for each $n \in N$ and $f \in \mathrm{Hom}_H(kG, N)$.

It turns out it is possible to generalize both the extension and coextension of scalars to arbitrary categories. Both the universal properties and the explicit construction have analogues in the categorical setting. We now explain how this is done.

2.2. **Kan extensions.** Let $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$ be functors.

Generalizing restriction of scalars is straightforward. Given a functor $H : \mathcal{D} \to \mathcal{E}$, we have the functor $K^* H = H \circ K : \mathcal{C} \to \mathcal{E}$. The assignment of $K^* H$ to H can be made into a functor $K^* : \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$ in the usual way.

When $\mathcal{C} = H$, $\mathcal{D} = G$ are groups, $\mathcal{E} = k$-**Mod** and $K = \alpha : H \to G$ is a functor[2], the functor K^* is precisely the restriction of scalars $\alpha^* : kG$-**Mod** $\to kH$-**Mod**.

How should we generalize the extension and coextension of scalars? We follow their description as adjoint functors.

Consider a category $\mathcal{H}$ where objects are pairs (H, γ) consisting of a functor $H : \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\gamma : F \Rightarrow HK$. A morphism in $\mathcal{H}$ from (H, γ) to (H', γ) is a natural transformation $\delta : H \Rightarrow H'$ such that $\delta K \circ \gamma = \gamma'$. Here δK is natural transformation obtained from δ by restricting along K on components and morphisms.

If the category $\mathcal{H}$ admits an initial object (H, γ), we shall call it a **left Kan extension** of F along K. More explicitly, a pair (H, γ) in $\mathcal{H}$ is a left Kan extension of F along K if for any other pair (H', γ') in $\mathcal{H}$, there is a unique natural transformation $\delta : H \Rightarrow H'$ such that

[2]This is exactly the same as a group homomorphism $H \to G$.

$\delta K \circ \gamma = \gamma'$. While left Kan extensions may not always exist, they are unique up to unique natural isomorphisms when they do exist by definition. We will write the left Kan extension of F along K by $\mathrm{Lan}_K F$ if it exists. We will see later that it is possible to impose conditions on the categories $\mathcal{C}$ and $\mathcal{E}$ so that every $F : \mathcal{C} \to \mathcal{E}$ admits a left Kan extension along K.

Suppose now that every $F : \mathcal{C} \to \mathcal{E}$ admits a left Kan extension along K. Then we can regard the assignment to each $F \in \mathcal{E}^{\mathcal{C}}$ its left Kan extension $\mathrm{Lan}_K F \in \mathcal{E}^{\mathcal{D}}$ as a functor $\mathrm{Lan} : \mathcal{E}^{\mathcal{C}} \to \mathcal{E}^{\mathcal{D}}$. Then the definition of left Kan extensions shows that Lan_K is left adjoint to K^*.

Now we see that in the case that $K = \alpha : H \to G$, then $\mathrm{Lan}_K : kH\text{-}\mathbf{Mod} \to kG\text{-}\mathbf{Mod}$ must be exactly $\alpha_!$ if the tentative functor Lan is defined. In fact, one can try to verify that our explicit construction for the extension of scalars directly by looking at its universal property. We will not do that here because this is an argument that generalizes to all categories, so we will later present the general version.

There is a dual story as always. We may associate to functors $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$ a category $\mathcal{H}'$. The objects are pairs (H, γ) where $H : \mathcal{D} \to \mathcal{E}$ is a functor and γ a natural transformation $HK \Rightarrow F$. A morphism in $\mathcal{H}'$ from (H', γ') to (H, γ) is a natural transformation $\delta : H' \Rightarrow H$ such that $\gamma \circ \delta K = \gamma'$. A **right Kan extension** of F along K is a final object of $\mathcal{H}'$ if it exists. We write the right Kan extension of F along K as $\mathrm{Ran}_K F$.

If $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are such that $\mathrm{Ran}_K F$ exists for any $F : \mathcal{C} \to \mathcal{E}$, then its universal property shows that Ran_K is right adjoint to K^*. For a homomorphism $\alpha : H \to G$, the right Kan extension is the coextension of scalars.

2.3. **Coend formula for left Kan extensions.** We had a lingering question from the last section. When do Kan extension always exist? We now give an answer to this question along with a formula for computing left Kan extensions.

Given a set S and an object $e \in \mathcal{E}$, we let $S \cdot e = \coprod_{s \in S} e$. Then $\cdot$ is made in an obvious way into a functor $\mathrm{Set} \times \mathcal{E} \to \mathcal{E}$.

Proposition 2.1. *Let $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$ be functors. Suppose that $\mathcal{C}$ is small, $\mathcal{D}$ is locally small and $\mathcal{E}$ is cocomplete. Then $\mathrm{Lan}_K F : \mathcal{D} \to \mathcal{E}$ exists and is computed by the formula*

$$\mathrm{Lan}_K F(d) \cong \int^{c \in \mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(Kc, d) \cdot Fc.$$

We defer the proof to the next section. For the rest of this section, we will spend some time making sense of the formula.

Remark 1. Before we try the formula on our ongoing example, let us think about why the various conditions on the categories are the sensible ones for the formula to work. We require $\mathcal{C}$ to be small to get a small diagram. The presence of the bifunctor $\mathrm{Hom}_{\mathcal{D}}(-,-)$ requires that $\mathcal{D}$ is locally small. Since coends are a type of colimit, we would like $\mathcal{E}$ to be cocomplete for their existence.

Back to the example. For clarity, we now write BG for the group G as a category. The expression $g \in G$ means that g is an element of the group G, or in other words, a morphism in the category BG. Consider a functor $\alpha : BH \to BG$ and a kH-module $V : BH \to k\text{-}\mathbf{Mod}$. Denote the unique object of BH, BG both as $*$. The left Kan extension of V evaluated at

$* \in G$ according to the formula is

$$\mathrm{Lan}_\alpha V \cong \int^{x \in BH} \mathrm{Hom}_{BG}(\alpha(x), *) \cdot V(x).$$

All three arguments in functor $\mathrm{Hom}_{BG}(\alpha(-), -) \cdot V(-)$ can only be evaluated at $*$. Its value in k-**Mod** is k-vector space $\bigoplus_{g \in G} V$. The coequalizer formula for coends 1.3 gives

$$\int^{x \in BH} \mathrm{Hom}_{BG}(\alpha(x), *) \cdot V(x) \cong \mathrm{coeq}\left(\bigoplus_{h \in H} \bigoplus_{g \in G} V \underset{b}{\overset{a}{\rightrightarrows}} \bigoplus_{g \in G} V \right).$$

Given $h \in H$ and $g \in G$, let us label the element $v \in V$ indexed by h, g in $\bigoplus_{h \in H} \bigoplus_{g \in G} V$ by (h, g, v). Similarly, we label the element $v \in V$ indexed by g in $\bigoplus_{g \in G} V$ by (g, v). The linear maps a and b then are defined concretely by the formulas $a(h, g, v) = (g, hv)$ and $b(h, g, v) = (g\alpha(h), v)$ respectively. Thus, we have

$$\mathrm{coeq}\left(\bigoplus_{h \in H} \bigoplus_{g \in G} V \underset{b}{\overset{a}{\rightrightarrows}} \bigoplus_{g \in G} V \right) \cong \left(\bigoplus_{g \in G} V \right) / \left\langle (g, hv) - (g\alpha(h), v) : g \in G, h \in H, v \in V \right\rangle.$$

The latter is of course just $kG \otimes_H V = \alpha_!(V)$. The left G-action on $\mathrm{Lan}_\alpha V = kG \otimes_H V$ comes from the middle argument of the functor $\mathrm{Hom}_{BG}(\alpha(-), -) \cdot V(-)$. This means $g \in G$ acts on $g' \otimes v$ the usual way as $g(g' \otimes v) = gg' \otimes v$.

Another way to frame the above discussion is to note that the functor $\mathrm{Hom}_{BG}(\alpha(-), -) \cdot V(-)$ is the k-vector space $\bigoplus_{g \in G} = kG \otimes_k V$ with three compactible actions: an action of H and G on the left as well as an action of H on the right via α. Then

$$\int^{x \in BH} \mathrm{Hom}_{BG}(\alpha(x), *) \cdot V(x) \cong kG \otimes_H V$$

as kG-modules follows from our discussion about cobalancing actions.

2.4. **The proof of 2.1.** Now that we have an idea how to think about the coend formula 2.1 for left Kan extensions, let us give a direct proof of it. The notation here will be the same as in the proposition. We begin with some set up.

The canonical map $\mathrm{Hom}_{\mathcal{D}}(Kx, d) \cdot Fx \to Hd$, which exists by virtue of Hd being a cowedge under $\mathrm{Hom}_{\mathcal{D}}(K-, d) \cdot F-$, will be written as ι_x. Although notation is ambiguous because it makes no reference to d, it will not cause any harm because it is easy to deduce which object in $\mathcal{D}$ is meant.

We will be working extensively with coproducts of the form $\mathrm{Hom}_{\mathcal{D}}(d', d) \cdot Fx$ where $d, d' \in \mathcal{D}$ and $x \in \mathcal{C}$. It will be convenient to denote the canonical map $Fx \to \coprod_{f : d' \to d} Fx = \mathrm{Hom}_{\mathcal{D}}(d', d) \cdot Fx$ at the component of $f : d' \to d$ by $[f]$. Here is an easy observation that will be used again and again.

Lemma 2.2. *Given morphisms $g : d \to d'', h : d''' \to d'$, we have the commutative triangles*

$$\begin{array}{ccc} Fx & \xrightarrow{[f]} & \mathrm{Hom}_{\mathcal{D}}(d', d) \cdot Fx \\ & \searrow^{[gf]} & \downarrow{\scriptstyle g_*} \\ & & \mathrm{Hom}_{\mathcal{D}}(d', d'') \cdot Fx \end{array} \qquad \begin{array}{ccc} Fx & \xrightarrow{[f]} & \mathrm{Hom}_{\mathcal{D}}(d', d) \cdot Fx \\ & \searrow^{[fg]} & \downarrow{\scriptstyle h^*} \\ & & \mathrm{Hom}_{\mathcal{D}}(d''', d') \cdot Fx \end{array}$$

Proof. This follows directly from the definition of the induced map between coproducts. □

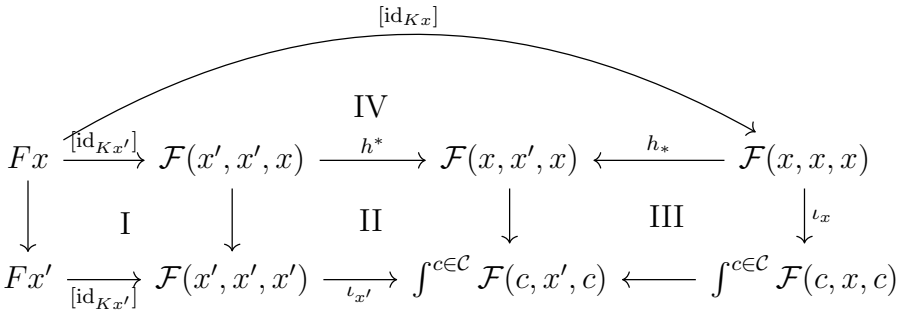
We break the the proof of 2.1 into two parts. By the remark after proposition 2.1, we know that the $H(-) = \int^{c\in\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(Kc, -) \cdot Fc$ is a well-defined functor $\mathcal{D} \to \mathcal{E}$.

We must first describe a natural transformation $\eta : F \Rightarrow H \circ K$. Since η is a necessry part of the definition of a left Kan extension, we note down the construction of η seperately for its own interest.

Lemma 2.3. *The map $\eta_x = \iota_x \circ [\mathrm{id}_{Kx}]$ is a natural transformation.*

Proof. Let us denote the trifunctor $\mathrm{Hom}_{\mathcal{D}}(K-, K-) \cdot F-$ by $\mathcal{F}$ to make the commutative diagrams below more readable.

Let $h : x \to x'$ be a morphism in $\mathcal{C}$. We have a commutative diagram



Square I commutes by definition of the induced map $\mathcal{F}(x', x', x) \to \mathcal{F}(x', x', x')$ on the coproducts. Square II commutes because $\int^{c\in\mathcal{C}} \mathcal{F}(c, x', c)$ is a cowedge under $\mathcal{F}$. Square III commutes by definition of the induced map on coends. Square IV commutes by 2.2.

This proves that η is a natural transformation. □

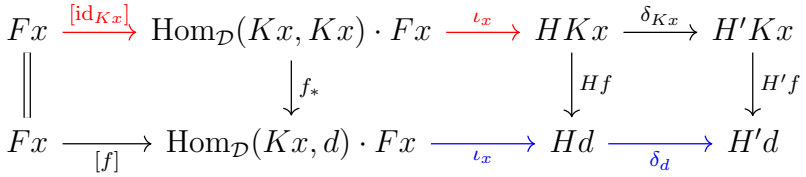
In our running example, the natural transformation above is precisely the unit of the adjunction pair $\alpha_!, \alpha^*$.

Now that we have constructed the pair (H, η), we can prove that it is the left Kan extension for F along K.

Proof of 2.1. We saw in 2.3 that $\eta : F \Rightarrow H \circ K$ is a natural transformation, so it remains to verify that (H, η) satisfies the universal property of left Kan extensions. The argument is of the standard kind.

Suppose that $(H'\eta')$ is another pair with $\eta' : F \Rightarrow H' \circ K$. First we show that any $\delta : H \Rightarrow H'$ such that $\delta K \circ \eta = \eta'$ must be unique, so let us assume the existence of δ for now.

Let $d \in \mathcal{D}$ and $x \in \mathcal{C}$. We have a diagram



The leftmost square commutes by lemma 2.2. The middle square commutes by definition of maps induced by H. The rightmost square commutes because δ is a natural transformation.

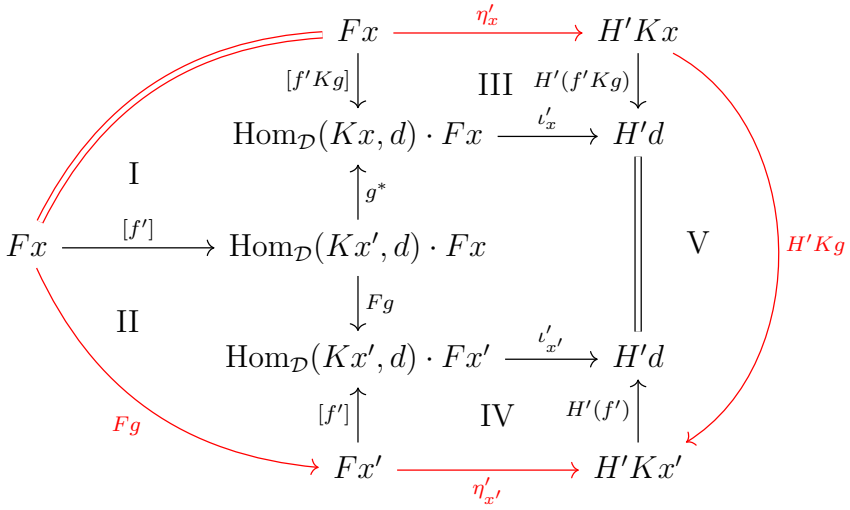
The red maps compose to η by definition, so we have $(\delta_d \circ \iota_d) \circ [f] = H'f \circ \eta'_x$. By definition of maps out of the coproduct $\mathrm{Hom}_{\mathcal{D}}(Kx, d) \cdot Fx$, the composition $\delta_d \circ \iota_x$ is uniquely determined by the diagram above.

To prove that δ itself is unique, we claim that any collection of morphism ι'_x such that the diagrams

$$(*) \qquad \begin{array}{ccc} Fx & \xrightarrow{\eta'_x} & H'Kx \\ {\scriptstyle [f]}\downarrow & & \downarrow{\scriptstyle H'f} \\ \mathrm{Hom}_{\mathcal{D}}(Kx, d) \cdot Fx & \xrightarrow[\iota'_x]{} & H'd \end{array}$$

commute makes $H'd$ a cowedge under $\mathrm{Hom}_{\mathcal{D}}(K-, d) \cdot F-$. The uniqueness of δ_x then follows by setting $\iota'_x = \delta_d \circ \iota_x$.

To see why the claim is true, suppose we have morphisms $g : x \to x'$ and $f' : Kx' \to d$ in $\mathcal{C}$ and $\mathcal{D}$ respectively. Consider the diagram



Square I commutes by lemma 2.2 again. The commutativity of square II and V are require no comment. Square III and IV are special cases of the diagram *, so are commutative by assumption. Finally the outer red "square" commutes because η' is natural. This proves the claim.

The existence of the appropriate δ actually also follows from claim we just made. We can use diagram * to define morphisms $\iota'_x : \mathrm{Hom}_{\mathcal{D}}(Kx, d) \cdot Fx \to H'd$. These morphisms make $H'd$ a cowedge $\mathrm{Hom}_{\mathcal{D}}(K-, d) \cdot F-$, so they induce a (unique) map $\delta_d : Hd \to Hd'$. We will not prove that the maps δ_d are natural because that is an argument similar to the ones presented here.

Finally, by letting $f = \mathrm{id}$ in diagram *, we see that $\delta K \circ \eta = \eta'$ as needed. □

2.5. **End formula for right Kan extensions.** There is a dual formula for right Kan extensions in terms of ends. We spell it out carefully because dualization can be confusing.

First, in what sense are right Kan extension dual to left Kan extensions? We make a simple observation about natural transformations under the process of dualization. Let $\mathcal{A}, \mathcal{B}$ be any two categories and $F, G : \mathcal{A} \to \mathcal{B}$. Then we may regard the functors as $F^{op}, G^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$. This is what people usually mean by "reversing all the arrows". Given a morphism $t : x \to y$

in $\mathcal{A}$, consider the two squares

$$\begin{array}{ccc} Fx & \xrightarrow{\alpha_x} & Gx \\ {\scriptstyle Ft}\downarrow & & \downarrow{\scriptstyle Gt} \\ Fy & \xrightarrow{\alpha_y} & Gy \end{array} \qquad \begin{array}{ccc} F^{op}x^{op} & \xleftarrow{\alpha_x^{op}} & G^{op}x^{op} \\ {\scriptstyle F^{op}t^{op}}\uparrow & & \uparrow{\scriptstyle G^{op}t^{op}} \\ F^{op}y^{op} & \xleftarrow{\alpha_y^{op}} & G^{op}y^{op} \end{array}$$

The left square is in $\mathcal{B}$ and the right square is the corresponding square in $\mathcal{B}^{op}$. Then the left square commute if and only if the right square commute. Thus, a natural transformation $\alpha : F \Rightarrow G$ is the same thing as a natural transformation $\alpha^{op} : G^{op} \Rightarrow F^{op}$. Symbolically, we can write this as $(-)^{op} : \mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(A, B) \cong \mathrm{Hom}_{\mathcal{B}^{op}\mathcal{A}^{op}}(G^{op}, F^{op})$.

From this, it follows that a right Kan extension of a functor $F : \mathcal{C} \to \mathcal{E}$ along a functor $K : \mathcal{C} \to \mathcal{D}$ is precisely the left Kan extension of the functor $F^{op} : \mathcal{C}^{op} \to \mathcal{E}^{op}$ along the functor $K^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$.

The same idea of flipping natural transformations around shows that the end of a bifunctor $\mathcal{F} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{E}$ is the coend of the bifunctor $\mathcal{F}^{op} : \mathcal{C}^{op} \times \mathcal{C} \cong (\mathcal{C}^{op} \times \mathcal{C})^{op} \to \mathcal{E}^{op}$. A warning here is that although we have an identification $\mathcal{C}^{op} \times \mathcal{C} \cong (\mathcal{C}^{op} \times \mathcal{C})^{op}$ which shows that $\mathcal{F}^{op}$ is a bifunctor, it does not even make sense to say that $\mathcal{F}$ is $\mathcal{F}^{op}$. For example, their codomain are $\mathcal{E}$ and $\mathcal{E}^{op}$ respectively, which are certainly not the same.

One final thing we need to dualize is the operation $- \cdot - : \mathrm{Set} \times \mathcal{E} \to \mathcal{E}$. To do this, note that given any set S and object $e \in \mathcal{E}$, we have by definition $S \cdot e = \coprod_{s \in S}(e) = \mathrm{colim}(S \to \mathcal{E})$. Here we regard S as a discrete category and the functor $S \to \mathcal{E}$ sends every object of S to $e \in \mathcal{E}$. If we were computing the coproduct $S^{op} \cdot e^{op}$ in $\mathcal{E}^{op}$, then the same reasoning with reversing natural transformation gives $S^{op} \cdot e^{op} = \mathrm{colim}(S^{op} \to \mathcal{E}^{op}) = \lim(S \to \mathcal{E}) = \prod_{s \in S} e$. We write e^S for $\prod_{s \in S} e$. It can be checked that e^S is a functor $\mathrm{Set}^{op} \times \mathcal{E} \to \mathcal{E}$. Moreover, the functor $- \cdot - : \mathrm{Set} \times \mathcal{E}^{op} \to \mathcal{E}^{op}$ correspond to $(-)^- : \mathrm{Set}^{op} \times \mathcal{E} \to \mathcal{E}$ when $(-)^{op}$ is applied.

We deduce the dual of 2.1 from the discussion above.

Proposition 2.4. *Let $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$ be functors. Suppose that $\mathcal{C}$ is small, $\mathcal{D}$ is locally small and $\mathcal{E}$ is complete. Then $\mathrm{Ran}_K F : \mathcal{D} \to \mathcal{E}$ exists and is computed by the formula*

$$\mathrm{Ran}_K F(d) \cong \int_{c \in \mathcal{C}} Fc^{\mathrm{Hom}_{\mathcal{D}}(d, Kc)}.$$

We now specialize to the case V is a kH-module and $\alpha : BH \to BG$ a group homomorphism regarded as a functor. By the end formula for right Kan extension, we have

$$\mathrm{Ran}_K F(*) \cong \int_{x \in BH} Fx^{\mathrm{Hom}_{BG}(*, \alpha(x))} = \mathrm{eq}\left(\prod_{g \in G} V \rightrightarrows \prod_{h \in H} \prod_{g \in G} V \right).$$

Note that when V is a k-module, the product $\prod_{g \in G} V$ can be identified with the k-module of k-linear maps $\mathrm{Hom}_k(kG, V)$. Under this identification, the equalizer consists of those $f \in \mathrm{Hom}_k(kG, V)$ such that $f(hg) = hf(g)$ for all $h \in H$ and $g \in G$. Thus, the equalizer is precisely $\mathrm{Hom}_H(kG, V)$, which is the coextension of scalars $\alpha_* V$.

3. Bonus: The Yoneda and coYoneda lemma

3.1. **The Yoneda lemma.** Using the end formula for right Kan extension, we can recover the Yoneda lemma very easily. If $F : \mathcal{C} \to \mathrm{Set}$ is a functor with $\mathcal{C}$ a small category, then we may consider the right Kan extension of F along the identity $\mathrm{id} : \mathcal{C} \to \mathcal{C}$. By inspection

of the universal properties involved, we have $F \cong \mathrm{Ran}_{\mathrm{id}_{\mathcal{C}}} F$. By the coend formula and proposition 1.1, we have

$$Fc \cong \mathrm{Ran}_{\mathrm{id}_{\mathcal{C}}} F \cong \int_{x \in \mathcal{C}} Fx^{\mathrm{Hom}_{\mathcal{C}}(c,x)} \cong \int_{x \in \mathcal{C}} \mathrm{Hom}_{\mathrm{Set}}(\mathrm{Hom}_{\mathcal{C}}(c,x), Fx) \cong \mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(c,-), F-)$$

which is natural in $c \in \mathcal{C}$.

The above computation reveals that the Yoneda lemma is like the ring theory fact that any R-module homomorphism $\phi : R \to M$, where R is a ring (with identity!) and M is a right R-module, is determined by the $\phi(1)$. The Yoneda lemma differs from the ring theory fact in two ways however. First of all, the Yoneda lemma as we have it here is unenriched. Secondly, there are many represented functors $\mathrm{Hom}_{\mathcal{C}}(c,-)$, so the Yoneda lemma is more like a many-object version of the ring theory fact.

Another way to interpret the Yoneda lemma is that it is analogous to saying that the coextension of scalar of a kG-module V via the identity $G \to G$ is just the kG-module V again.

3.2. **The coYoneda lemma.** The nice thing about deriving the Yoneda lemma using our Kan extension technology is that it suggests making use of the dual formula as well. By considering the left Kan extension of $F : \mathcal{C} \to \mathrm{Set}$ along the identity $\mathrm{id} : \mathcal{C} \to \mathcal{C}$, which is also just F itself, we arrive at the formula

$$Fc \cong \int^{x \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(x,c) \cdot Fc.$$

This is called the coYoneda lemma for the obvious reason. It is also called the density theorem because it also expresses the fact that every presheaf is a colimit of representable presheaves. For example, this means every simplicial set is obtained by gluing standard simplices. We will not make that precise here.

Like with the Yoneda lemma, this can be seen as an analogue of the fact that applying the extension of scalars for a kG-module V along $\mathrm{id}_G : G \to G$ is again the kG-module V.

References

[1] Emily Riehl. **Categorical homotopy theory**, volume 24. Cambridge University Press Cambridge, 2014.